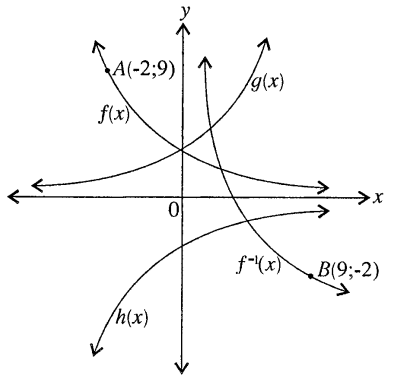
12.4 *Inverse of the exponential function is the* log-*function.*

$\therefore\ f^{-1} = \log_{\frac{1}{3}} x \qquad [f(x) = \frac{1}{3}^{x}]$

12.5 $f(x) = \frac{1}{3}^{x}$

Decending function : $0 < \frac{1}{3} < 1$
Horizontal asymptote : $y = 0$
y-intercept : $y = 1$



Sketch $g(x)$ *by reflecting* $f(x)$ *in the y-axis*

Sketch $h(x)$ *by reflecting* $f(x)$ *in the x-axis*

Sketch $f^{-1}(x)$ *by reflecting* $f(x)$ *in the line* $y = x$

12.6 $A(-2;9)$ *is a point on* $f(x)$. *Therefore,* $B(9;-2)$ *is a point on* $f^{-1}(x)$.
See sketch.

$\therefore\ f^{-1}(x) > -2$ *if* $0 < x < 9$

EXERCISE 3

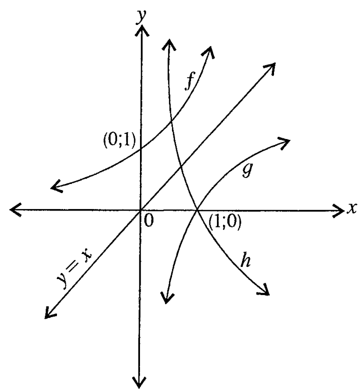
1. Solve for x, correct to two decimal digits :

1.1 $8^x = 160$ 1.2 $3^x = 15$

1.3 $3.6^x = 10$ 1.4 $7^x = 126(5^x)$

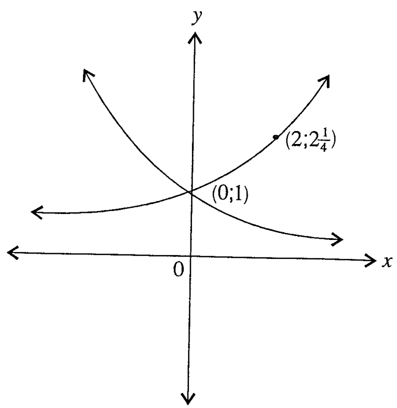
1.5 $2^x.4^{x+1} = 24$ 1.6 $35(1,25)^x = 245$

2. In the sketch, the following functions are represented :
   * f, with equation $y = 3^x$
   * g, the reflection of f in the line $y = x$. (so g is the inverse of f.)
   * h, the reflection of g in the x-axis.



2.1 Determine the defining equations of g and h in the form $y =$........
2.2 Determine, with the aid of the sketch, the value(s) of x for which :
a) $3^x > 0$
b) $\log_{\frac{1}{3}} x \leq 0$

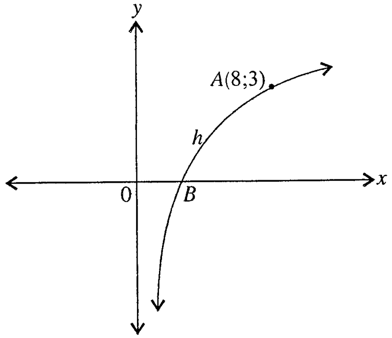
3. In the sketch f represents the function $f(x) = a^x$, $(a > 0)$. h is symmetrical to f about the y-axis. The point $(2 ; 2\frac{1}{4})$ lies on the curve of f.



3.1 Find the value of a.
3.2 If $g(x) = f^{-1}(x)$, the inverse of f, determine the equation of g.
3.3 Write down the domain of g.
3.4 Determine the equation of h.
3.5 Write down the range of h.

(Solutions p. 254-255)

4. The figure shows the graph of $h(x) = \log_a x$. The point $A(8;3)$ lies on the curve of h and B is the x-intercept of h.

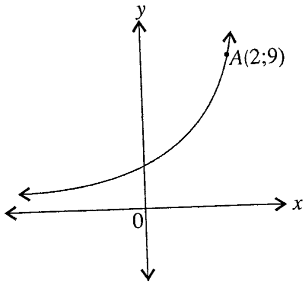


4.1 Determine the value of a.
4.2 Write down the coordinates of B.
4.3 Determine the equation of h^{-1}, the inverse of h, in the form $h^{-1}(x) = \ldots\ldots\ldots$
4.4 Determine the value of $h^{-1}(3)$.
4.5 Write down the domain of h.
4.6 Determine the equation of $g(x)$, the reflection of $h(x)$ about the y-axis.

5. Given : $g(x) = \log_a x$. $C(\frac{1}{2};-1)$, is a point on the curve of g.

5.1 Calculate the value of a.
5.2 Give the equation of the function h which is symmetrical to g with respect to the x-axis.
5.3 Sketch the graphs of $g(x)$ and $h(x)$ on the same system of axes.
5.4 For which value of x is $g(x) = h(x)$?
5.5 Use the graph to solve for x : $\log_{\frac{1}{2}} x = 1$.

6. The sketch shows the graph of $f(x) = a^x$. $A(2;9)$ is a point on the curve of f.



6.1 Determine the value of a.
6.2 Find the equation of $h(x)$ if h is the reflection of f in the y-axis.
6.3 Determine the equation of $f^{-1}(x)$, the inverse of $f(x)$.
6.4 Find the equation of $g(x)$, the reflection of $f^{-1}(x)$ in the y-axis.
6.5 For which values of x is $f(x) \geq 9$?

(Solutions p. 255-256)

FINANCIAL MATHEMATICS

You have already used the following formulae in Grade 11 :

> Simple interest formula : $A = P(1 \pm in)$
>
> Compound interest formula : $A = P(1 \pm i)^n$
>
> *where* A = *the future value of the money invested*
> P = *the initial amount invested*
> i = *interest rate, expressed as a decimal fraction*
> n = *number of interest bearing periods*

It is a good idea first to revise the sections of Financial Mathematics in the Grade 10 and Grade 11 study guides.

1. Calculation of the investment period (n) in the formula $A = P(1 \pm i)^n$

To determine the value of n in the formula $A = P(1 \pm i)^n$, we will make use of logarithms.

Consider the following examples.

Example 1

Lesiba was awarded a bonus of R4 500 for excellence at his workplace. He invested this money at a rate of 10% per annum, compounded annually, and received R9 646,12 at the end of the period. For how many years was the money invested?

Solution

Compound interest $\therefore A = P(1 + i)^n$

$\therefore\ 9646,12 = 4500(1+0,1)^n$ $\quad$ $[A = 9646,12 : P = 4500 : i = 0,1]$

$\therefore\ \frac{9646,12}{4500} = (1+0,1)^n$ $\quad$ [*Divide by* 4500]

$\therefore\ \frac{9646,12}{4500} = (1,1)^n$ $\quad$ [*Simplify inside brackets on RHS*]

$\therefore\ \log\frac{9646,12}{4500} = \log(1,1)^n$ $\quad$ [*Take logs on both sides*]

$\therefore\ \log\frac{9646,12}{4500} = n\log 1,1$ $\quad$ $[\log a^n = n\log a]$

$$\therefore\ n = \frac{\log\frac{9646,12}{4500}}{\log 1,1}$$

$\therefore\ n = 7,9999$

$\therefore$ *The money was invested for* 8 *years*

Example 2

A man invests R15 000 at 12% per annum, compounded monthly. At the end of the period he receives R27 250. For what period did he invest the money?

Solution

$A = (1+i)^n$

$\therefore\ 27250 = 15000(1+\frac{0,12}{12})^{12n}$ $\quad$ [*Interest compounded monthly* : *divide i by* 12 *and multiply n by* 12]

$\therefore\ \frac{27250}{15000} = \left(1+\frac{0,12}{12}\right)^{12n}$ $\quad$ [*Divide by* 15000]

$\therefore\ \frac{109}{60} = (1,01)^{12n}$ $\quad$ $[\frac{27250}{15000} = \frac{109}{60}$ *and* $1+\frac{0,12}{12} = 1,01]$

$\therefore\ \log\frac{109}{60} = \log(1,01)^{12n}$ $\quad$ [*Take logs on both sides*]

$\therefore\ \log\frac{109}{60} = 12n\log(1,01)$ $\quad$ $[\log a^n = n\log a]$

$$\therefore\ 12n = \frac{\log\frac{109}{60}}{\log 1,01}$$

$\therefore\ 12n = 59,9979$

$\therefore\ n = 4,9998$ $\quad$ [*Divide by* 12]

$\therefore$ *The money was invested for* 5 *years*

Example 3

Office equipment depreciates at 16% per annum on the reducing balance method. How long will it take before the value of the office equipment is halved?

Solution

Reducing balance depreciation $\therefore\ A = P(1-i)^n$

Let $P = 2x$. *The money halves, therefore* $A = x$

$\therefore\ x = 2x(1-0,16)^n$

$\therefore\ \frac{x}{2x} = (1-0,16)^n$ $\quad$ [*Divide by* $2x$]

$\therefore\ 0,5 = (0,84)^n$

$\therefore\ \log(0,5) = \log(0,84)^n$ $\quad$ [*Take logs on both sides*]

$\therefore\ \log(0,5) = n\log(0,84)$ $\quad$ $[\log a^n = n\log a]$

$\therefore\ n = \frac{\log(0,5)}{\log(0,84)}$ $\quad$ [*Divide by* $\log 0,84$]

$\therefore\ n = 3,9755$

$\therefore$ *The value will be halved after* 4 *years*

2. Future value of an annuity

When people save money for a specific reason, they do not always invest a single initial amount, but deposit fixed monthly amounts into a savings account or an investment fund. Compound interest is paid on the money deposited into the fund.

The future value of an annuity is the sum of all the deposits made, plus all the interest earned.

Consider the following examples.

Example 4

Peter deposits R500 at the beginning of each month into a savings account. The interest rate on the savings account is 12% p.a., compounded monthly. What will the total balance of his savings account be after six months?

Solution

The interest rate is 12% *p.a, compounded monthly, therefore divide i by* 12.
On the first deposit he will recieve interest for 6 *months.*

$\therefore\ 500(1+0,01)^6 = 500(1,01)^6 \qquad [\frac{i}{12} = \frac{0,12}{12} = 0,01]$

On the second deposit he will recieve interest for 5 *months* : $500(1,01)^5$
On the last deposit he will recieve interest for one month : $500(1,01)$

Let's place the deposits Peter made on a number line.

0	1	2	3	4	5	6 months
$500(1,01)^6$	$500(1,01)^5$	$500(1,01)^4$	$500(1,01)^3$	$500(1,01)^2$	$500(1,01)$	

Lets start at the last payment.
The total amount available after 6 *months will be* :

$$500(1,01)+500(1,01)^2+500(1,01)^3+500(1,01)^4+500(1,01)^5+500(1,01)^6$$

This is a geometric series with :

$a = 500(1,01)$ *and* $r = (1,01) \qquad [r = \frac{T_2}{T_1} = \frac{500(1,01)^2}{500(1,01)} = 1,01]$

Now calculate the sum of the first 6 *terms.*

$$S_n = \frac{a(r^n-1)}{r-1}$$

$$\therefore\ S_6 = \frac{500(1,01)[(1,01)^6-1]}{1,01-1}$$

$$= \underline{R3\ 106,77}$$

Example 5

R1 000 is paid into a savings account at the end of each year. The interest rate is 12% per annum, compounded annually. What will the account balance be after 4 years?

The first deposit was made at the end of the first year and the last deposit at the end of the fourth year.

Solution

Let's use the number line again.

0	1	2	3	4	*Years*
	$R1000$	$R1000$	$R1000$	$R1000$	

The first amount was invested at the end of the first year, therefore for 3 *years.*

The first deposit earns interest for 3 *years* : $\quad \therefore\ A = 1000(1+0,12)^3$
The second deposit earns interest for 2 *years* : $\quad \therefore\ A = 1000(1+0,12)^2$
The third deposit earns interest for 1 *year* : $\quad \therefore\ A = 1000(1+0,12)$
The last deposit was only made at the end of the investment period and therefore earns no interest : $\quad \therefore\ A = 1000$

Now add all the amounts :

$1000 + 1000(1,12) + 1000(1,12)^2 + 1000(1,12)^3 \qquad [(1+0,12) = 1,12]$

This is a geometric series with :
$a = 1000$ *and* $r = (1,12)$ [*Each term is multiplied by* 1,12]

To calculate the total amount after 4 *years, we can use the formula for the sum*

of a geometric series : $S_n = \dfrac{a(r^n - 1)}{r-1}$

$$\therefore\ S_4 = \frac{1000(1,12^4 - 1)}{1,12 - 1} \qquad [n = 4]$$

$$\therefore\ \underline{S_4 = R4779,33}$$

Note the difference between example 4 and example 5.

Example 4 : The first payment was made at the beginning of the period and the last payment was made one month before the end of the period.
Example 5 : The first payment was made at the end of the first month and the last payment made at the end of the period.

The following formula is useful when we want to calculate the future value of an annuity or a savings plan :

The future value annuity formula

$$F = \frac{x[(1+i)^n - 1]}{i}$$

where F = *Future value*
x = *fixed payments per period*
i = *interest rate, expressed as a decimal fraction*
n = *number of payments*

However, this formula can only be used if a **final payment, that does not earn any interest, is made at the end of the period**, as in example 5.

If the final payment is not made at the end of the investment period and it does earn interest, calculate the future value by using the formula for the sum of a geometric series as in example 4.

Example 6

Chris decides to provide for his retirement at the age of 60 by depositing R1 000 each month into an investment account. The interest rate on this account is 11,5% per annum, compounded monthly. If he makes his first payment on his 40^{th} birthday and the final payment one month before his 60^{th} birthday, what will the balance of his investment account be on his 60^{th} birthday?

Solution

The last payment is not made at the end of the period, therefore use the formula for the sum of a geometric series.

Final payment $= 1000(1 + \dfrac{0,115}{12})$ [*Earns interest for one month*]

Second last payment $= 1000(1 + \dfrac{0,115}{12})^2$ [*Earns interest for two months*]

$\therefore$ *Geometric series* : $a = 1000(1 + \frac{0,115}{12})$ *and* $r = (1 + \frac{0,115}{12})$

First payment on his fortieth birthday, final payment one month before his sixtieth birthday $\therefore$ 20×12 *payments* $= 240$. *Therefore calculate* S_{240}

$$S_{240} = \frac{a(r^{240} - 1)}{r - 1}$$

$$= \frac{1000(1 + \frac{0,115}{12})((1 + \frac{0,115}{12})^{240} - 1)}{(1 + \frac{0,115}{12}) - 1} \qquad [a = 1000(1 + \tfrac{0,115}{12}) \text{ and } r = (1 + \tfrac{0,115}{12})]$$

$$= \underline{R933\ 966,61}$$

Example 7

Robert decides to deposit R500 per month, for a period of 5 years, into an investment fund at an interest rate of 12% p.a., compounded monthly. He makes his first payment at the end of the first month, and the final payment at the end of the 5 year period. Calculate the value of his investment after 5 years.

Solution

First payment was made at the end of the first month and the final payment was made at the end of the period. Therefore, use the future value annuity formula :

$$F = \frac{x[(1+i)^n - 1]}{i}$$

$$\therefore\ F = \frac{500[(1+0,01)^{60} - 1]}{0,01} \qquad [x = 500;\ \textit{interest compounded monthly} \ \therefore\ \tfrac{i}{12} = \tfrac{0,12}{12} = 0,01;\ n = 12 \times 5 = 60]$$

$$= \frac{500[(1,01)^{60} - 1]}{0,01} \qquad [\textit{Simplify } (1+0,01) \textit{ to } 1,01, \textit{ it makes the calculations easier}]$$

$$= \underline{R40\ 834,83}$$

Example 8

A man starts saving money to buy a car. He opens a savings account and immediately deposits R1 000. After that he deposits R1 000 each month and continues to do so for the next three years. The interest rate is 13% p.a., compounded monthly. How much money will he have after 3 years to buy a car?

Solution

Now the first payment is made immediately and not at the end of the first month.
When the first payment is made immediately and the last payment is made at the end of the period, you may use the future value annuity formula.
But remember :The first payment was made immediately, therefore the total number of payments is $3 \times 12 +$ *the first payment that was made.*

$$F = \frac{x[(1+i)^n - 1]}{i}$$

$$\therefore\ F = \frac{1000[(1+\frac{0,13}{12})^{37} - 1]}{\frac{0,13}{12}} \qquad [x = 1000;\ \textit{interest compounded monthly} \ \therefore\ \tfrac{i}{12} = \tfrac{0,13}{12};\ n = 3 \times 12 + 1 = 37]$$

$$= \underline{R45\ 217,23}$$

Example 9

At the end of each month Elaine deposits an amount into a savings account. The interest rate is 12% per annum, compounded monthly. What amount must she deposit each month in order to have R50 000 in her account after 8 years? The final payment is made at the end of the 8-year period.

Solution

Payments is made at the end of each month, the first payment made at the end of the first month and the last payment at the end of the period.

Therefore, use the future value annuity formula and determine x.

$$F = \frac{x[(1+i)^n - 1]}{i}$$

$$\therefore\ 50000 = \frac{x[(1+0,01)^{96} - 1]}{0,01} \qquad [F = 5000;\ \textit{interest compounded monthly} \ \therefore\ \tfrac{1}{12} = \tfrac{0,12}{12} = 0,01;\ n = 96]$$

$$\therefore\ 50000 \times 0,01 = x[(1,01)^{96} - 1] \qquad [\textit{Multiply by } 0,01]$$

$$\therefore\ \frac{50000 \times 0,01}{[(1,01)^{96} - 1]} = x \qquad [\textit{Now divide by } [(1,01)^{96} - 1] \textit{ to find } x]$$

$$\therefore\ x = 312,64$$

<u>R312,64 must be saved each month</u>

Example 10

At the end of each month Jean deposits R1 000 into a savings account. If the interest rate on the savings account is 10% per annum, compounded monthly, how long will it take before he has R50 000 in his savings account?

Solution

Use the future value annuity formula and determine n.

$$F = \frac{x[(1+i)^n - 1]}{i}$$

$$\therefore\ 50000 = \frac{1000[(1+\frac{0,1}{12})^n - 1]}{\frac{0,1}{12}} \qquad [F = 50000;\ x = 1000;\ \textit{compounded monthly}\ \therefore\ \tfrac{i}{12} = \tfrac{0,1}{12};\ \textit{determine } n]$$

$$\therefore\ 50000 \times \frac{0,1}{12} = 1000[(\tfrac{121}{120})^n - 1] \qquad [\textit{Multiply by } \tfrac{0,1}{12};\ (1+\tfrac{0,1}{12}) = \tfrac{121}{120}]$$

$$\therefore\ \frac{50000 \times \frac{0,1}{12}}{1000} = (\tfrac{121}{120})^n - 1 \qquad [\textit{Divide by } 1000]$$

$$\therefore\ \frac{50000 \times \frac{0,1}{12}}{1000} + 1 = (\tfrac{121}{120})^n \qquad [\textit{Move } -1 \textit{ to the left-hand side}]$$

$$\therefore\ \tfrac{17}{12} = (\tfrac{121}{120})^n \qquad [\textit{Simplify left-hand side}]$$

$$\therefore\ \log\tfrac{17}{12} = \log(\tfrac{121}{120})^n \qquad [\textit{Take logs on both sides}]$$

$$\therefore\ \log\tfrac{17}{12} = n\log\tfrac{121}{120} \qquad [\log a^n = n\log a]$$

$$\therefore\ n = \frac{\log\frac{17}{12}}{\log\frac{121}{120}} \qquad [\textit{Divide by } \log\tfrac{121}{120}]$$

$$\therefore\ n = 41,97$$

$\underline{\therefore\ \textit{It will take } 42 \textit{ months or } 3\tfrac{1}{2} \textit{ years}}$

Note : n = number of payments made. Monthly payments are made, therefore if $n = 41,97$ it will take 42 months.

Example 11

Gregory needs R200 000 in 4 years' time to buy a new car.

11.1 Calculate how much he will need to pay monthly if the interest rate is 8% per annum, compounded monthly.

11.2 Twelve months after Gregory made his first deposit and every 12 months thereafter, he withdraws R10 000 from his account. What will the new monthly deposit be if he makes 4 such withdrawels?

Solution

11.1 $F = \dfrac{x[(1+i)^n - 1]}{i}$

$$\therefore\ 200000 = \frac{x[(1+\frac{0,08}{12})^{48} - 1]}{\frac{0,08}{12}} \qquad [F = 200000;\ \textit{interest compounded monthly, therefore}\ i \div 12;\ n = 48]$$

$$\therefore\ \frac{200000 \times \frac{0,08}{12}}{[(1+\frac{0,08}{12})^{48} - 1]} = x \qquad [\textit{Multiply by } \tfrac{0,08}{12} \textit{ and divide by } [(1+\tfrac{0,08}{12})^{48} - 1]]$$

$\underline{\therefore\ x = R3\ 549,25}$

11.2 *He withdraws R10 000 every year.*

Calculate the equal monthly payments that he will need to make in order to have R10 000 after one year. Therefore, use the future value annnuity formula with $F = 10000$, $i = \frac{0,08}{12}$ and $n = 12$. Remember, he wants R10000 available the end of the year, therefore 12 monthly payments.

$$F = \frac{x[(1+i)^n - 1]}{i}$$

$$\therefore\ 10000 = \frac{x[(1+\frac{0,08}{12})^{12} - 1]}{\frac{0,07}{12}} \qquad [F = 10000;\ \textit{divide } i \textit{ by } 12,\ 12 \textit{ payments in } 1 \textit{ year, therefore } n = 12]$$

$$\therefore \frac{10000 \times \frac{0,08}{12}}{[(1+\frac{0,08}{12})^{12}-1]} = x$$ *[Multiply by* $\frac{0,08}{12}$ *and divide by* $[(1+\frac{0,08}{12})^{48}-1]$]

$\therefore x = R803,22$

$\therefore$ *New monthly deposit* $= R3549,25 + R803,22 = \underline{R4\ 352,47}$

> *Remember : n = number of payments made*
> *The first payment is usually made at the end of the first month or first time interval. However, if a payment is made immediately, remember to add that payment.*

3. Sinking funds

A sinking fund is money that a company sets aside in order to make provision for a large capital expense, or to replace equipment after a certian period of time. A sinking fund is simply a savings plan or a future value annuity.

Example 12

A school buys a bus for R650 000. The value of the bus decreases according to a reducing balance method of depreciation at 18% per annum. The school wants to buy a new bus after 5 years.
The price of the bus increases by 15% per year, compounded annually.

12.1 What will the value of the bus be after 5 years?

12.2 What will a new bus cost in 5 years time?

12.3 Calculate the value of the sinking fund that the school will require in order to replace the old bus.

12.4 If the interest rate is 12% per annum, compounded monthly, what amount will the school need to deposit monthly into the sinking fund to be able to afford the new bus?

Solution

12.1 *Reducing balance depreciation* $\therefore A = P(1-i)^n$

$\therefore A = 650000(1-0,18)^5$

$\therefore A = \underline{R240\ 980,90}$

12.2 *The price of the bus increases* $\therefore A = P(1+i)^n$

$\therefore A = 650000(1+0,15)^5$

$= \underline{R1\ 307\ 382,17}$

12.3 *Sinking fund = price of new bus – value of old bus*

$= R1\ 307\ 382,17 - R240\ 980,9$

$= \underline{R1\ 066\ 401,27}$

12.4 *Use the formula* : $F = \frac{x[(1+i)^n - 1]}{i}$ *and calculate* x.

$$\therefore 1066401,27 = \frac{x[(1+0,01)^{60}-1]}{0,01}$$ [$F = 1066401,27$; $i \div 12$, *therefore* $\frac{0,12}{12} = 0,01$; $n = 60$]

$$\therefore 1066401,27 \times 0,01 = x[(1,01)^{60}-1]$$ [*Multiply by* $0,01$; $(1+1,01) = 1,01$]

$$\therefore \frac{1066401,27 \times 0,01}{[(1,01)^{60}-1]} = x$$ [*Divide by* $[(1,01)^{60}-1]$]

$\therefore R13057,49 = x$

$\therefore$ *The monthly payment* $= \underline{R13\ 057,49}$

4. Present value of an annuity or loan repayments

A sum of money is normally borrowed from a financial institution to buy a house, a car, furniture, etc. The load is repayed by fixed monthly repayments.
As a result of the effect of compound interest, the full amount you eventually pay back is a lot more than the initial amount you borrowed.
The present value of an annuity is the value of the loan before interest has been added.

Example 13

A loan is granted to Jane at an interest rate of 24% per annum, compounded monthly. She repays the loan in 6 monthly payments of R2 000 per month. Calculate the amount of the loan granted to her.

Solution

Let's make use of a time line again :

1	2	3	4	5	6 months
R2000	R2000	R2000	R2000	R2000	R2000

The first payment of R2000 includes interest. We must therefore first determine what the initial amount is before interest. We use the formula :

$$A = P(1+i)^n$$

$\therefore\ 2000 = P(1+0{,}02)$ [*Interest compounded monthly* $\therefore\ \frac{0{,}24}{12} = 0{,}02;\ 1$ *month* $\therefore\ n = 1$]

$\therefore\ \dfrac{2000}{(1{,}02)} = P$ $[(1+0{,}02) = 1{,}02]$

$\therefore\ 2000(1{,}02)^{-1} = P$ $[\frac{1}{1{,}02} = (1{,}02)^{-1}]$

The payment made in the second month is :

$2000 = P(1+0{,}02)^2$ $\therefore\ P = \dfrac{2000}{(1{,}02)^2}$ $\therefore\ P = 2000(1{,}02)^{-2}$

And again, the payment made in the third month is : $P = 2000(1{,}02)^{-3}$

Therefore, the amount that was borrowed is :

$$2000(1{,}02)^{-1} + 2000(1{,}02)^{-2} + 2000(1{,}02)^{-3} + \ldots\ldots\ldots\ldots + 2000(1{,}02)^{-6}$$

which is a geometric series with $a = 2000(1{,}02)^{-1}$ *and* $r = (1{,}02)^{-1}$

Calculate the sum of the first 6 terms : $S_6 = \dfrac{2000(1{,}02)^{-1}(1-(1{,}02^{-1})^6)}{1-(1{,}02)^{-1}}$

$$= \frac{2000(1{,}02)^{-1}(1-1{,}02)^{-6})}{1-(1{,}02)^{-1}}$$

$$= \underline{\underline{11202{,}86}}$$

The present value annuity formula

$$P = \frac{x[1-(1+i)^{-n}]}{i}$$

where : P = *present value of the loan (The amount borrowed)*
x = *fixed repayments*
i = *interest rate, expressed as a decimal fraction*
n = *number of payments*

Example 14

Philip takes out a bank loan to buy a car. What is the largest loan amount he can qualify for if he can afford to repay R1 200 per month over 3 years? The first payment is due at the end of the first month after the loan has been granted and the interest rate is 24% per annum, compounded monthly.

Solution

Now use the present value annuity formula and calculate P.

$$P = \frac{x[1-(1+i)^{-n}]}{i}$$

$$\therefore\ P = \frac{1200[1-(1+0,02)^{-36}]}{0,02} \qquad [x = R1200;\ \tfrac{i}{12} = \tfrac{0,24}{12} = 0,02,\ n = 36\ (3 \times 12)]$$

$$\therefore\ P = R30\ 586,61 \qquad \therefore\ \textit{He can qualify for a loan of } R30\ 586,61$$

Example 15

Colene borrows R30 000 at an interest rate of 22% per annum, compounded monthly. What will her monthly repayments be if she pays back the loan over 4 years?

Solution

$$P = \frac{x[1-(1+i)^{-n}]}{i}$$

$$\therefore\ 30000 = \frac{x[1-(1+\frac{0,22}{12})^{-48}}{\frac{0,22}{12}} \qquad [P = R30000;\ \tfrac{i}{12} = \tfrac{0,22}{12},\ n = 4 \times 12 = 48;\ \textit{now calculate } x]$$

$$\therefore\ 30000 \times \tfrac{0,22}{12} = x[1-(1+\tfrac{0,22}{12})^{-48}] \qquad [\textit{Multiply by } \tfrac{0,22}{12}]$$

$$\therefore\ \frac{30000 \times \frac{0,22}{12}}{[1-(1+\frac{0,22}{12})^{-48}]} = x \qquad [\textit{Divide by } [1-(1+\tfrac{0,22}{12})^{-48}]]$$

$$\therefore\ 945,18 = x \qquad \therefore\ \textit{The monthly repayment is } R945,18$$

Example 16

A man wants to borrow R50 000 from a bank. The interest rate is 18% p.a., compounded monthly. If the man can afford monthly instalments of R1 200, how long will it take to repay the loan in full?

Solution

Use the present value annuity formula and calculate n.

$$P = \frac{x[1-(1+i)^{-n}]}{i}$$

$$\therefore\ 50000 = \frac{1200[1-(1+0,015)^{-n}]}{0,015} \qquad [P = 50000;\quad x = 1200;\ \tfrac{i}{12} = \tfrac{0,18}{12} = 0,015;\ \textit{calculate } n]$$

$$\therefore\ \frac{50000 \times 0,015}{1200} = [1-(1+1,015)^{-n}] \qquad [\textit{Multiply by } 0,015,\ \textit{divide by } 1200]$$

$$\therefore\ \frac{50000 \times 0,015}{1200} - 1 = -(1,015)^{-n} \qquad [\textit{Move } 1 \textit{ to } LHS,\ (1+0,015 = 1,015)]$$

$$\therefore\ -0,375 = -(1,015)^{-n} \qquad [\textit{Simplify left-hand side}]$$

$$\therefore\ 0,375 = (1,015)^{-n} \qquad [\textit{Multiply by } -1]$$

$$\therefore\ \log 0,375 = \log(1,015)^{-n} \qquad [\textit{Take logs on both sides}]$$

$$\therefore\ \log 0,375 = -n\log(1,015) \qquad [\textit{Remember}: \log a^n = n\log a]$$

$$\therefore\ \frac{\log 0,375}{\log 1,015} = -n \qquad \therefore\ -n = -65,87 \qquad \therefore\ n = 65,87$$

$\therefore$ *It will take 66 months or 5½ years to repay the loan*

Example 17

Ernest buys a car for R240 000. He pays a deposit of 20% and takes out a bank loan for the balance at a rate of 11,5% per annum, compounded monthly. Calculate:

17.1 The value of the loan.
17.2 The monthly repayments if the loan is repaid over 5 years.
17.3 The balance on the loan at the end of two years immediately after Ernest made the 24^{th} payment.

Solution

17.1 *Value of loan* $= R240\,000 - deposit$
$= R240\,000 - (\frac{20}{100} \times 240000)$ [*Pays* 20% *deposit*]
$= R240\,000 - R48\,000$
$= \underline{R192\,000}$

17.2 $P = \dfrac{x[1-(1+i)^{-n}]}{i}$ [*Use present value annuity formula*]

$\therefore\ 192000 = \dfrac{x[1-(1+\frac{0,115}{12})^{-60}]}{\frac{0,115}{12}}$ [$P = 192000$; *interest compounded monthly, therefore* $\frac{i}{12}$ *and* $n = 60$]

$\therefore\ 192000 \times \dfrac{0,115}{12} = x[1-(1+\dfrac{0,115}{12})^{-60}$ [*Multiply by* $\frac{0,115}{12}$]

$\therefore\ \dfrac{192000 \times \frac{0,115}{12}}{[1-(1+\frac{0,115}{12})^{-60}} = x$ [*Divide by* $[1-(1+\frac{0,115}{12})^{-60}]$]

$\therefore\ x = 4222,58$

$\therefore$ $\underline{\textit{Monthly repayments} = R4\,222,58}$

17.3 *We can calculate the outstanding balance in two different ways* :

A short method is to determine the outstanding repayments after two years and then use the present value annuity formula. He took out the loan for 5 years, therefore there were still 36 repayments outstanding.

$P = \dfrac{x[1-(1+i)^{-n}]}{i}$

Outstanding balance $= \dfrac{4222,58[1-(1+\frac{0,115}{12})^{-36}]}{\frac{0,115}{12}}$ [36 *repayments outstanding*]

Outstanding balance $= \underline{R128\,050,07}$

Outstanding balance = (*Loan* + *Interest*) − (*Monthly repayments* + *Interest*)

To calculate the loan plus interest, use compound interest formula. The outstanding balance after two years has to be calculated, therefore $n = 2 \times 24$.

Loan + *Interest* $= P(1+i)^n$

$= 192000(1+\frac{0,115}{12})^{24}$ [$P = 192000,\ n = 24$]

$= 241386,72$

To calculate the monthly repayments after two years, we use the future value annuity formula.

Monthly repayments + *Interest* $= \dfrac{x[(1+i)^n - 1]}{i}$ [$x = 4222,58,\ n = 24$]

$= \dfrac{4222,58[(1+\frac{0,115}{12})^{24} - 1]}{\frac{0,115}{12}}$

$= 113336,61$

$\therefore$ *Balance* $= 241386,72 - 113336,61 = \underline{R128\,050,11}$

Although there is a small difference between the two answers, both answers will be accepted.

Example 18

Vanessa took out a loan from a bank for R8 000. She agrees to pay monthly instalments of R750. The bank charges interest at 12% p.a., compounded monthly.

18.1 Determine the number of payments required to settle the loan.
18.2 Determine the value of the last payment made to settle the loan.

Solution

18.1 *Remember* : n = *the number of payments. Therefore, we divide the interest rate by* 12 *but do not multiply n by* 12.

$$P = \frac{x[1-(1+i)^{-n}]}{i}$$

$$\therefore 8000 = \frac{750[1-(1+\frac{0,12}{12})^{-n}]}{\frac{0,12}{12}}$$ [*We calculate the number of payments, therefore we do not multiply n by* 12]

$$\therefore 8000 \times 0,01 = 750[1-(1+0,01)^{-n}] \qquad [\tfrac{0,12}{12} = 0,01]$$

$$\therefore \frac{8000 \times 0,01}{750} = [1-(1+0,01)^{-n}] \qquad [\textit{Divide by } 750]$$

$$\therefore \frac{8000 \times 0,01}{750} - 1 = -(1+0,01)^{-n}$$

$$\therefore -0,89333\ldots = -(1,01)^{-n}$$

$$\therefore 0,893333.. = (1,01)^{-n} \qquad [\textit{Multiply by } -1]$$

$$\therefore \log 0,89333\ldots = -n \log 1,01 \qquad [\textit{Take logs on both sides}]$$

$$\therefore -n = \frac{\log 0,89333}{\log 1,01}$$

$$\therefore -n = -11,335\ldots$$

$$\therefore n = 11,335$$

$\therefore$ 12 *payments will be made*

18.2 *First calculate the balance after the* 11^{th} *payment, using the second method in* 17.3.

Outstanding balance = (*Loan* + *Interest*) − (*Monthly repayments* + *Interest*)

$$\textit{Outstanding balance} = P(1+i)^n - \frac{x[(1+i)^n - 1]}{i} \qquad [\textit{See example } 17.3]$$

$$\therefore \textit{Balance} = 8000(1+\tfrac{0,12}{12})^{11} - \frac{750[(1+\frac{0,12}{12})^{11} - 1]}{\frac{0,12}{12}}$$

$$\therefore \textit{Balance} = 8000(1,01)^{11} - \frac{750[(1,01)^{11} - 1]}{0,01} \qquad (1 + \tfrac{0,12}{12} = 0,01]$$

$$= R250,22$$

Remember, Vanessa will make the last payment at the end of the 12^{th} *month, therefore she will pay interest on the balance for one month.*

$\therefore$ *Final payment* $= 250,22(1+0,01) \qquad \underline{= R252,72}$

Note : *Vanessa did not make* 12 *equal monthly repayments but* 11 *repayments of R*750 *and a last repayment of* R252,72. *Therefore we cannot use the present value annuity formula like we did in the first method in* 17.3.

*The monthly repayments on a loan of R*8 000 *at an interest rate of* 12% *p.a is* :

$$8000 = \frac{x[1-(1+\frac{0,12}{12})^{-12}]}{\frac{0,12}{12}} \qquad \therefore x = \frac{8000 \times 0,01}{[1-(1+0,01)^{12}]} \qquad = R710,79.$$

*She payed R*39,21 *more each month. If she payed* 710,79 *monthly, then the outstanding balance after* 11 *payments would be more.*

Example 19

On his retirement Sheldon recieves R900 000. He invests the money at an interest rate of 8,5% p.a., compounded monthly. He plans to withdraw R10 000 at the end of each month. The first withdrawal will be made at the end of the first month. For how many months will he be able to withdraw this amount?

Solution

When an investment is made and monthly withdrawals take place, it can be seen as a loan which the bank will repay at Rx per month. Therefore, when monthly withdrawals are made from an investment, use the present value annuity formula with
P = the amount invested and x = the monthly withdrawels

$$P = \frac{x[1-(1+i)^{-n}]}{i}$$

$$\therefore\ 900000 = \frac{10000[1-(1+\frac{0{,}085}{12})^{-n}]}{\frac{0{,}085}{12}}$$ [*R900 000 is invested, monthly withdrawels of R10 000*]

$$\therefore\ \frac{900000 \times \frac{0{,}085}{12}}{10000} = 1-(1+\tfrac{0{,}085}{12})^{-n}$$ [*Multiply by* $\frac{0{,}085}{12}$, *divide by* 10000]

$$\therefore\ \frac{900000 \times \frac{0{,}085}{12}}{10000} - 1 = -(1+\tfrac{0{,}085}{12})^{-n}$$ [*Move 1 to LHS*]

$$\therefore\ -\frac{29}{80} = -(1+\tfrac{0{,}085}{12})^{-n}$$ [*Simplify LHS*]

$$\therefore\ \frac{29}{80} = (1+\tfrac{0{,}085}{12})^{-n}$$ [*Multiply by* -1]

$$\therefore\ \log\frac{29}{80} = \log(1+\tfrac{0{,}085}{12})^{-n}$$ [*Take logs on both sides*]

$$\therefore\ \log\frac{29}{80} = -n\log(1+\tfrac{0{,}085}{12})$$

$$\therefore\ -n = \frac{\log\frac{29}{80}}{\log(1+\frac{0{,}085}{12})}$$

$$\therefore\ -n = -143{,}76$$

$$\therefore\ n = 143{,}76$$

$\therefore$ *For* 143 *months*

5. **Comparing investment and loan options**

It is always a good idea to compare the different options available before you borrow money or invest in an annuity, so that you can make an informed decision.

Example 20

You want to buy a car and need a loan of R100 000. You apply for a loan and the bank offers you the following two options:

Option 1 : A loan at an interest rate of 24% per annum, compounded monthly, repayable in equal monthly repayments over a period of 4 years. The first payment is made at the end of the first month.

Option 2 : A loan at an interest rate of 22% per annum, compounded monthly, repayable in equal monthly repayments over a period of 5 years, the first payment being made at the end of the first month.

20.1 For each option, calculate the following :

20.1.1 the monthly repayments.
20.1.2 the total amount repaid.

20.2 Which option is the best option?
Give reasons for your answer.

Solution

20.1.1 Option 1 :

$$P = \frac{x[1-(1+i)^{-n}]}{i}$$

$$\therefore\ 100000 = \frac{x[1-(1+0,02)^{-48}]}{0.02} \qquad [P = 100000,\ \tfrac{i}{12} = \tfrac{0,24}{12} = 0,02,\ n = 48]$$

$$\therefore\ 100000 \times 0,02 = x[1-(1,02)^{-48}] \qquad [\textit{Multiply by } 0,02]$$

$$\therefore\ \frac{100000 \times 0,02}{[1-(1,02)^{-48}]} = x \qquad [\textit{Divide by } 1-(1,02)^{-48}]$$

$$\therefore\ x = 3260,18 \qquad \therefore\ R3\,260,18 \textit{ per month.}$$

Option 2:

$$P = \frac{x[1-(1+i)^{-n}}{i}$$

$$\therefore\ 100000 = \frac{x[1-(1+\frac{0,22}{12})^{-60}}{\frac{0,22}{12}} \qquad [P = 100000;\ \tfrac{i}{12} = \tfrac{0,22}{12};\ n = 60]$$

$$\therefore\ 100000 \times \tfrac{0,22}{12} = x[1-(\tfrac{611}{600})^{-60}] \qquad [1+\tfrac{0,22}{12} = \tfrac{611}{600}]$$

$$\therefore\ \frac{100000 \times \frac{0,22}{12}}{[1-(\frac{611}{600})^{-60}]} = x$$

$$\therefore\ x = 2761,89$$

$\therefore$ *R*2 761,89 *per month*

20.1.2 Option 1 $\quad 3260,18 \times 48 = R156\,488,64$
Option 2 $\quad 2761,89 \times 60 = R165\,714,40$

20.2 *Option* 1 *offers the best option as R*9 224,76 *less is paid back.*

6. Micro loans

Micro lenders offer short-term loans at very high interest rates. Many people are forced to use micro lenders when they need cash urgently and they do not qualify for a loan at formal institutions for some or other reason.
The interest is calculated over the full period of the loan at simple interest rate.
Currently, strict legislation governs micro lending to protect both those granting and those making the loans.
It is very important that you clearly understand any loan agreement you sign, and that you compare it with all other options available before making a decision.

7. Pyramid schemes

Pyramid schemes are forbidden in terms of the law. In a pyramid scheme, a large number of people invest at the bottom of the pyramid, and the money goes to a few people at the top of the pyramid. The broker usually promises you a return of thousands of Rands when your name reaches the top of the pyramid. In reality, the pyramid usually collapses before your name reaches the top. Many people have lost thousands of Rands on pyramid schemes.

When is which formula applied?

A single payment is made for a fixed period

at simple interest : $A = P(1 + in)$

at compound interest : $A = P(1 + i)^n$

Depreciation takes place

on a straight line basis : $A = P(1 - in)$

on a reducing balance basis : $A = P(1 - i)^n$

Save by making regular equal payments

Future value annuity formula : $F = \dfrac{x[(1 + i)^n - 1]}{i}$

Remember : n = *number of payments made. The first payment is usually made at the end of the first month. However, if a first payment is made immediately, use the future value annuity formula, just add that first payment.*

When an amount is withdrawn from an investment at the end of each year, use the future value annuity formula and calculate the monthly payments to be made in order to have that amount available at the end of the year. In this case $n = 12$. *See example* 12.

Loan repayments

Present value annuity formula : $P = \dfrac{x[1 - (1 + i)^{-n}]}{i}$

Outstanding balance on a loan

Outstanding balance = (loan + interest) – (monthly repayments + interest)

$$= P(1 + i)^n - \frac{x[(1 + i)^n - 1]}{i}$$

or

Use the present value annuity formula with n = *the number of outstanding repayments. However, this method can only be used if equal monthly repayments is made.*

See examples 17 *and* 18.

A fixed amount is withdrawn monthly from an investment

Consider it as a loan repayable by the bank and use the formula :

$$P = \frac{x[1 - (1 + i)^{-n}]}{i}$$

See example 19.