14. A woman wants to start her own business. She borrows R250 000 at an interest rate of 12% per annum, compounded monthly, for a period of 5 years. After 2 years she repays R120 000 to the bank, and after another year a further sum of R80 000. What amount will she have to pay the bank at the end of the 5 year period?

15. Leonard invests R20 000 at an interest rate of 12% p.a., compounded monthly. After 3 years the interest rate changes to 14% per annum, compounded quarterly. Calculate the value of the investment after 5 years.

16. When the nominal interest rate is 12% per annum, compounded monthly, what is the effective interest rate?

17. A businessman can invest his money at Bank A at 13% per annum, compounded monthly, and at Bank B at 15% per annum, compounded quarterly. Calculate the effective interest rate in each case. Which bank offers the best investment?

18. If the effective interest rate is 13,8% on an investment, what is the nominal interest rate if the interest is compounded monthly?

(Solutions pg. 263)

FUNCTIONS AND GRAPHS

1. The linear function defined by $y = ax + q$

The graph of $y = ax + q$ has already been dealt with in Grade 9 and 10. Revise functions and graphs in the Grade 10 Study guide.

2 The Parabola defined by $y = a(x+p)^2 + q$ $(y = ax^2 + bx + c)$

In Grade 10 you have sketched the graph of $y = ax^2 + q$. The parabola was symmetrical about the y-axis.

The effect of a : a is the vertical stretching or shrinking of the graph.

If a is positive, the function has a minimum value .

If a is negative, the function has a maximum value .

The effect of p and q

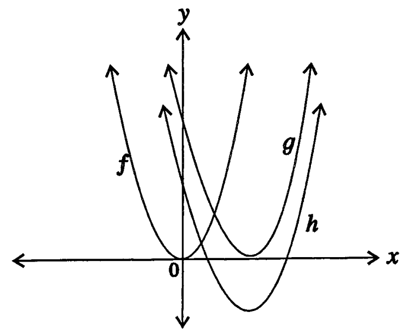
The function $f(x) = a(x+p)^2 + q$ is symmetrical about the line $x = -p$ and the coordinates of the turning point is $(-p; q)$.

The figure shows the graphs of

$f(x) = x^2$,

$g(x) = (x-2)^2$ and

$h(x) = (x-2)^2 - 2$



Thus : p is the **horizontal shift** of the graph p units to the left or to the right and q is the **vertical shift** of the graph q units upwards or downwards.

2.1 Sketching the parabola

To sketch the parabola, you must calculate the following

The x-intercepts : Let $y = 0$ and solve the equation.

The y-intercepts : Let $x = 0$ (this is always the c-value if the equation is in the form $y = ax^2 + bx + c$.)

The axis of symmetry and the coordinates of the turning point

This can be done in two different ways.

Equation of the axis of symmetry : $x = -\dfrac{b}{2a}$
This is also the x-**coordinate** of the turning point.
To find the y-**coordinate** of the turning point, substitute this x-value into the equation.

or

Write the equation in the form $y = a(x+p)^2 + q$ by completing the square.
Equation of the axis of symmetry : $x = -p$.
The coordinates of the turning point are $(-p;q)$ $\quad \therefore\ x = -p;\ y = q$

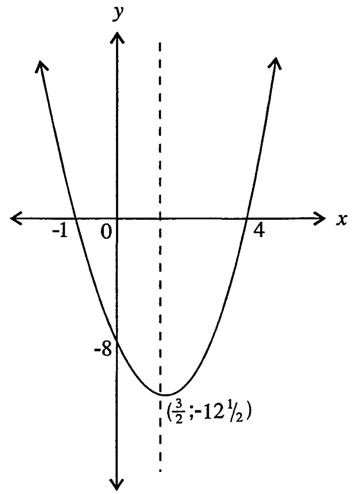
Example 1

Sketch the graph of $y = 2x^2 - 6x - 8$

Solution

y-intercept : $y = -8$ $\quad$ [*Let* $x = 0 \rightarrow y = 2(0)^2 - 6(0) - 8 = -8$]

x-intercept : *Let* $y = 0$



$\therefore\ 2x^2 - 6x - 8 = 0$
$\therefore\ x^2 - 3x - 4 = 0$ $\quad$ [*Divide by* 2.]
$\therefore\ (x-4)(x+1) = 0$
$\therefore\ x = 4$ *or* $x = -1$.

Turning point: $x = -\dfrac{b}{2a}$

$\therefore\ x = -\dfrac{-6}{2(2)} = \dfrac{6}{4} = \dfrac{3}{2}$

Substitute $x = \frac{3}{2}$ *into equation*

$\therefore\ y = 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) - 8$

$= -12\frac{1}{2}$ $\qquad \therefore$ *Turning point* $= (\frac{3}{2};-12\frac{1}{2})$

or

Write $y = 2x^2 - 6x - 8$ *in the form* $y = a(x+p)^2 + q$ *by completing the square.*

$y = 2x^2 - 6x - 8$

$\therefore\ y = 2(x^2 - 3x - 4)$ $\quad$ [*Divide by* 2]

$\therefore\ y = 2(x^2 - 3x + (-\frac{3}{2})^2 - (-\frac{3}{2})^2 - 4)$ $\quad$ [*Add and subtract* $(-\frac{3}{2})^2$]

$\therefore\ y = 2[(x - \frac{3}{2})^2 - \frac{9}{4} - 4]$ $\quad$ [*Factorise first three terms and square* $-(-\frac{3}{2})$]

$\therefore\ y = 2[(x - \frac{3}{2})^2 - \frac{25}{4}]$ $\quad$ [*Add last two terms*]

$\therefore\ y = 2(x - \frac{3}{2})^2 - \frac{25}{2}$ $\quad$ [*Multiply* $-\frac{25}{4}$ *by* 2 *in front of bracket* $\therefore\ 2(-\frac{25}{4}) = -\frac{25}{2}$]

$\therefore$ *Coordinates of turning point* : $\quad (\frac{3}{2};-\frac{25}{2}) = (\frac{3}{2};-12\frac{1}{2})$

Example 2

Draw a neat sketch graph of $y = -2(x-1)^2 + 8$.

Solution

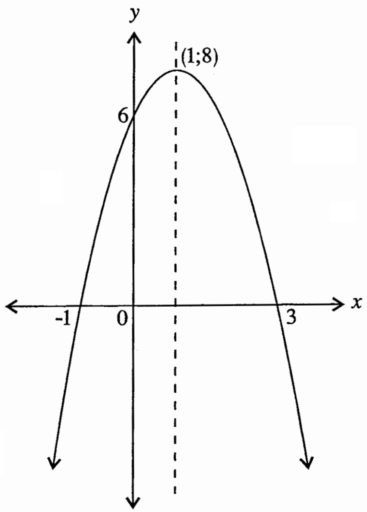
Equation is already in the form $y = a(x+p)^2 + q$

$\therefore$ *Coordinates of Turning Point*: $(-p;q)$ $\therefore$ $(-(-1);8) = (1;8)$

Axis of symmetry : $x = 1$

To find the x-and y-intercepts, first simplify the equation.

$y = -2(x-1)^2 + 8.$
$\therefore\ y = -2(x^2 - 2x + 1) + 8$
$\therefore\ y = -2x^2 + 4x - 2 + 8$
$\therefore\ y = -2x^2 + 4x + 6$



y-intercept : $y = 6$

x-intercepts : *Let* $y = 0$

$\therefore\ -2x^2 + 4x + 6 = 0$
$\therefore\ x^2 - 2x - 3 = 0$ [*Divide by* -2]
$\therefore\ (x-3)(x+1) = 0$
$\therefore\ x = 3$ *or* $x = -1$

2.2 Determining the equation of the parabola

When you are asked te find the equation of the parabola, one of the following equations are used

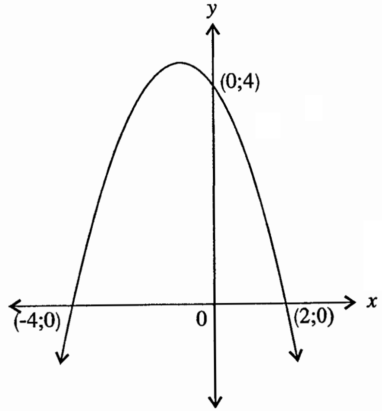
The roots and another point are given : $y = a(x - 1st\ root)(x - 2nd\ root)$

Turning Point and another point are given : $y = a(x+p)^2 + q$

Example 3

The diagram shows the graph of $y = ax^2 + bx + c$.

Find the values of a, b *and* c.



Solution

Roots are given :

$y = a(x - 1st\ root)(x - 2nd\ root)$

$\therefore\ y = a(x-2)(x-(-4))$ [*Roots* : 2 *and* -4]

$\therefore\ y = a(x-2)(x+4)$

$\therefore\ 4 = a(0-2)(0+4)$ [$(0;4)$ *is a point on the graph. Now let* $x = 0$ *and* $y = 4$ *and determine* a]

$\therefore\ 4 = a(-2)(4)$

$\therefore\ 4 = -8a$

$\therefore\ a = \frac{4}{-8}$

$\therefore\ a = -\frac{1}{2}$

Now substitute a with $-\frac{1}{2}$ *in* $y = a(x-2)(x+4)$ *and simplify.*

$\therefore\ y = -\frac{1}{2}(x-2)(x+4)$

$\therefore\ y = -\frac{1}{2}(x^2 + 2x - 8)$

$\therefore\ y = -\frac{1}{2}x^2 - x + 4$

$a = -\frac{1}{2},\ b = -1,\ c = 4$

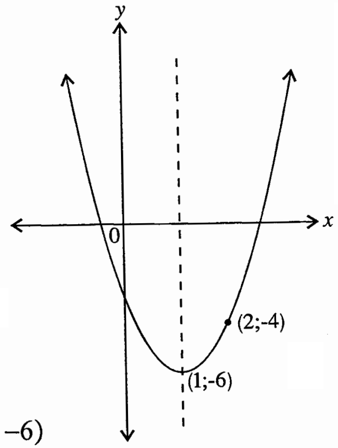
Example 4

In the diagram

$f(x) = ax^2 + bx + c$.

Determine the values

of a, b and c.



Solution

Coordinates of turning point : $(1;-6)$

$\therefore\ q = -6$ [*Coordinates of turning point* : $(-p;q)$]

$\therefore\ p = -1$ [*Note* : *Coordinates of turning point* : $(-p;q)$
$\therefore\ -p = 1 \quad \therefore\ p = -1$]

Turning point is given $\rightarrow y = a(x+p)^2 + q$.

Substitute p with -1 *and q with* -6.

$\therefore\ y = a(x-1)^2 - 6$
$\therefore\ -4 = a(2-1)^2 - 6$ [*To determine a, let* $x = 2$ *and* $y = -4$]
$\therefore\ -4 = a(1)^2 - 6$
$\therefore\ -4 + 6 = a$
$\therefore\ 2 = a$

Now let $a = 2$ *in* $y = a(x-1)^2 - 6$

$\therefore\ y = 2(x-1)^2 - 6$
$\therefore\ y = 2(x^2 - 2x + 1) - 6$
$\therefore\ y = 2x^2 - 4x + 2 - 6$
$\therefore\ y = 2x^2 - 4x - 4$.

$a = 2,\ b = -4,\ c = -4$

3. The hyperbola defined by $y = \dfrac{a}{x+p} + q$

The effect of *a*

If $a > 0$ (positive) the graph lies in the first and third quadrant.
If $a < 0$ (negative) the graph lies in the second and fourth quadrant.
a is the vertical shrinking or stretching of the graph.

The effect of *q*

The equation of the **horizontal asymptote** of the hyperbola is $y = q$.

q is the **vertical shift** of the graph q units upwards or downwards.

The effect of *p*

Division by 0 is undefined, therefore $x + p \neq 0$.
Thus, the equation of the **vertical asymptote** is $x = -p$ ($x + p = 0$ when $x = -p$).

p is the **horizontal shift** of the graph p units to the left or to the right.

3.1 Sketching the graph of the hyperbola

To sketch the hyperbola, you have to calculate the following :

The **horizontal asymptote** : The horizontal asymptote is the line $y = q$.

The **vertical asymptote** : The vertical asymptote is the line $x = -p$

The ***x*-intercept** : Let $y = 0$ and solve the equation.

The ***y*-intercept** : Let $x = 0$ and solve the equation.

If a is positive → graph in quadrant 1 and 3.

If a is negative → graph in quadrant 2 and 4.

Example 5

Sketch the graph of $y = \dfrac{12}{x+1} + 2$

Solution

Horizontal asymptote : $y = 2$

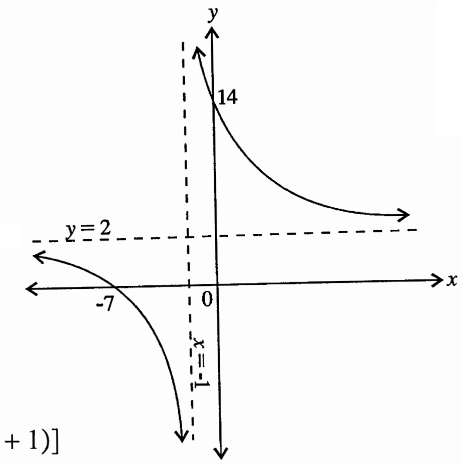
Vertical asymptote : $x = -1$ [*Let* $x + 1 = 0,\ \therefore\ x = -1$]

y-intercept: *Let* $x = 0$

$\therefore\ y = \dfrac{12}{0+1} + 2$

$\therefore\ y = 12 + 2$

$\therefore\ y = 14$



x-intercept: *Let* $y = 0$

$\therefore\ \dfrac{12}{x+1} + 2 = 0$

$\therefore\ 12 + 2(x+1) = 0$ [*Multiply by* $(x+1)$]

$\therefore\ 12 + 2x + 2 = 0$

$\therefore\ 2x = -14$

$\therefore\ x = -7$

$a > 0$, *graph in* 1*st and* 3*rd quadrant.*

Example 6

Sketch the graph of $y = \dfrac{-8}{x-2} - 1$

Solution

Horizontal asymptote : $y = -1$

Vertical asymptote : $x = 2$

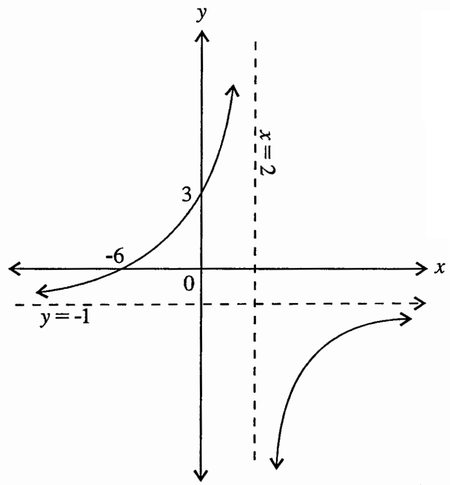
y-intercept : $x = 0$

$\therefore\ y = \dfrac{-8}{0-2} - 1$

$\therefore\ y = \dfrac{-8}{-2} - 1$

$\therefore\ y = 4 - 1$

$\therefore\ y = 3$



x-intercept : $y = 0$

$\therefore\ \dfrac{-8}{x-2} - 1 = 0$

$\therefore\ -8 - (x-2) = 0$

$\therefore\ -8 - x + 2 = 0$

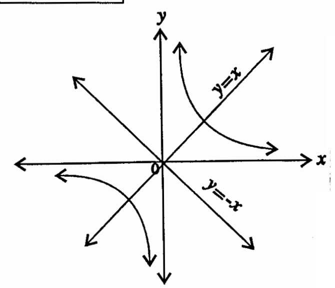
$\therefore\ -x = 6$

$\therefore\ x = -6$ $a < 0$, *therefore graph in* 2*nd and* 4*th quadrant.*

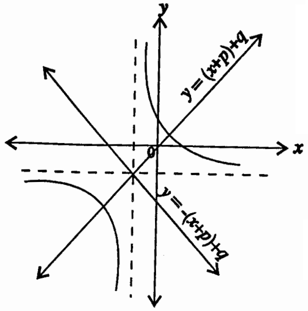
3.2 The axes of symmetry of the hyperbola

The function $f(x) = \frac{a}{x}$ is symmetrical about the line $y = x$ and the line $y = -x$.

Therefore, the equation of the axes of symmetry are $y = x$ and $y = -x$.



The equation of the axes of symmetry of the hyperbola defined by $f(x) = \dfrac{a}{x+p} + q$ are $y = (x+p)+q$ and $y = -(x+p)+q$.



Example 7

Determine the equation of the axes of symmetry of

7.1 $y = \dfrac{12}{x+1} + 2$

7.2 $y = \dfrac{-8}{x-2} - 1$

Solution

7.1 *Equation of axes of symmetry*

$$y = (x+p)+q \quad and \quad y = -(x+p)+q$$

$y = (x+1)+2$ *and* $y = -(x+1)+2$

$\therefore\ y = x+3$

$\therefore\ y = -x-1+2$

$\therefore\ y = -x+1$

$\therefore$ *Equation of axes of symmetry* : $\underline{y = x+3 \quad and \quad y = -x+1}$

7.2 $y = (x+p)+q \quad and \quad y = -(x+p)+q$

$y = (x-2)-1$ *and* $y = -(x-2)-1$

$\therefore\ y = x-2-1$

$\therefore\ y = x-3$

$\therefore\ y = -x+2-1$

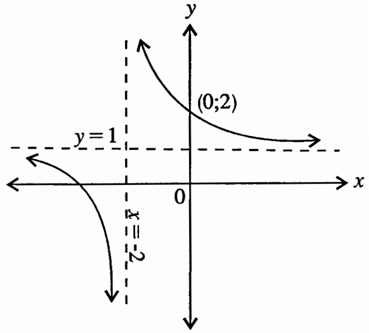
$\therefore\ y = -x+1$

$\underline{\therefore\ y = x-3 \quad and \quad y = -x+1}$

3.3 Determining the equation of the hyperbola

Example 8

The sketch represents the graph of $y = \dfrac{a}{x+p} + q$.



Find the values of a, p *and* q.

Solution

Vertical asymptote : $x = -2$

$\therefore\ -p = -2$ [*Vertical asymptote* : $x = -p$]

$\therefore\ p = 2$

Horizontal asymptote : $y = 1$

$\therefore\ q = 1$ [*Horizontal asymptote* : $y = q$]

Substitute $p = 2$ *and* $q = 1$ *into equation* .

$\therefore\ y = \dfrac{a}{x+2} + 1$

$\therefore\ 2 = \dfrac{a}{0+2} + 1$ [*Substitute x with* 0 *and y with* 2 *and determine a*]

$\therefore\ 2(2) = a+2$ [*Multiply by* 2]

$\therefore\ 4-2 = a$

$\therefore\ a = 2$

$\underline{\therefore\ a = 2\ ;\ p = 2\ ;\ q = 1}$

4. The exponential function defined by $y = a.b^{x+p} + q$

The effect of a : a is the vertical stretching or shrinking of the graph.

The effect of the base, b : The function is ascending if $b > 1$ and descending if $0 < b < 1$.

The effect of q : You already know that q is the **vertical shift** of the graph. The equation of the **horizontal asymptote** is $y = q$.
The graph of the exponential function do not have a vertical asymptote.

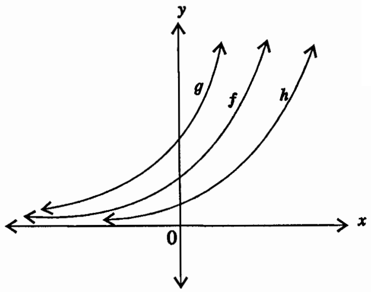
The effect of p

Sketched are the graphs of

$f(x) = 2^x$,

$g(x) = 2^{x+1}$ *and*

$h(x) = 2^{x-1}$.



Therefore, p is a **horizontal shift** of the graph. If $p > 0$, the graph moves to the left and if $p < 0$, the graph moves to the right.

4.1 Sketching the graph of the exponential function

To sketch the graph of the exponential function , calculate

The horizontal asymptote : The horizontal asymptote is always $y = q$.

The y-intercept : Let $x = 0$ and solve the equation.

The x-intercept : Let $y = 0$ and solve the equation. Note : If $q > 0$ there will be no x-intercept.

If the base >1 the graph will be ascending and if the base < 1 the graph will be descending.

Example 9

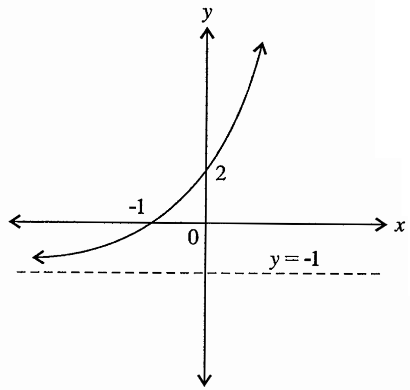
Sketch the graph of $y = 3^{x+1} - 1$

Solution

Horizontal asymptote : $y = -1$

y-intercept : *Let* $x = 0$

$\therefore\ y = 3^{0+1} - 1$
$\therefore\ y = 3 - 1$
$\therefore\ y = 2$



x-intercept : *Let* $y = 0$

$\therefore\ 3^{x+1} - 1 = 0$
$\therefore\ 3^{x+1} = 1$
$\therefore\ 3^{x+1} = 3^0$ [*Remember* : $1 = 3^0$]
$\therefore\ x + 1 = 0$ [*Bases are equal, equate exponents*]
$\therefore\ x = -1$

Base > 1 $\therefore$ *function is ascending.*

Example 10

Draw a sketch graph of $y = 2(\frac{1}{3})^{x-1} + 1$.

Solution

Horizontal asymptote : $y = 1$

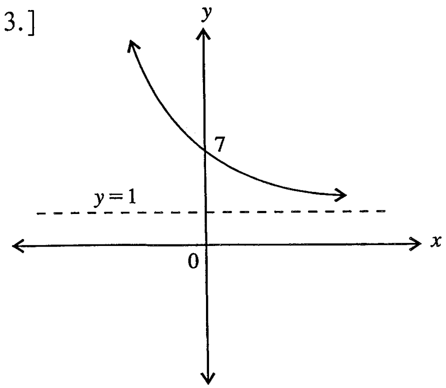
y-intercept : *Let* $x = 0$

$\therefore\ y = 2(\frac{1}{3})^{0-1} + 1$

$\therefore\ y = 2(\frac{1}{3})^{-1} + 1$

$\therefore\ y = 2(3) + 1 \qquad [(\frac{1}{3})^{-1} = 3.]$

$\therefore\ y = 7$



x-intercept : *Let* $y = 0$

$\therefore\ 2(\frac{1}{3})^{x-1} + 1 = 0$

$\therefore\ (\frac{1}{3})^{x-1} = -\frac{1}{2}$

$\therefore$ *no solution*

The horizontal asymptote : $y > 0 \quad \therefore$ *no x-intercept.*

Base $< 1 \ \therefore$ *function is decending.*

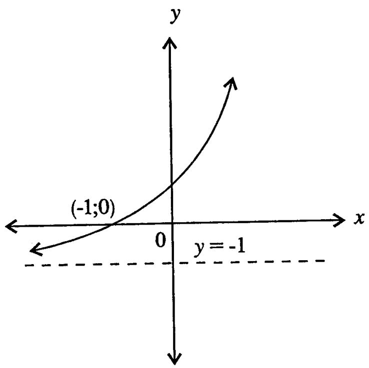
4.2 Determining the equation of the graph

Example 11

The diagram shows the

graph of $y = 2^{x+p} + q$

Determine the values of p and q.



Solution

Horizontal asymptote : $y = -1$

$\therefore\ q = -1 \qquad$ [*Horizontal asymptote is always* $y = q$]

To determine p, let $q = -1$ *and substitute* $(-1;0)$ *into the equation.*

$\therefore\ 0 = 2^{-1+p} - 1$

$\therefore\ 1 = 2^{-1+p}$

$\therefore\ 2^0 = 2^{-1+p} \qquad$ [*Equalise bases* $\rightarrow 2^0 = 1$]

$\therefore\ 0 = -1 + p \qquad$ [*Bases are equal, equate exponents*]

$\therefore\ p = 1$

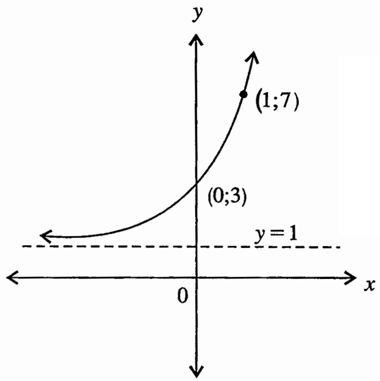
$\therefore\ p = 1\ ;\ q = -1$

Example 12

In the diagram the graph of

$y = a.b^x + q$ is shown.

Determine the equation of the graph.



Solution

Horizontal asymptote : $y = 1$

$\therefore\ q = 1 \qquad$ [*Horizontal asymptote always* $y = q$]

$\therefore\ y = a.b^x + 1 \qquad$ [*Substitute q with* 1]

Two points on the graph are given $\rightarrow (0;3)$ *and* $(1;7)$.

First substitute$(0;3)$ *into equation* :

$\therefore\ 3 = a(b)^0 + 1 \qquad [x = 0;\ y = 3\ ]$

$\therefore\ 3 - 1 = a(1) \qquad$ [*Remember* $b^0 = 1$]

$\therefore\ 2 = a$

Now let $a = 2$ *in* $y = a.b^x + 1$ *and substitute* $(1;7)$ *into equation.*

$\therefore\ y = 2b^x + 1$

$\therefore\ 7 = 2(b)^1 + 1$ [*Let* $x = 1$ *and* $y = 7$.]

$\therefore\ 7 - 1 = 2b$

$\therefore\ 6 = 2b$

$\therefore\ b = 3$

Equation of graph : $y = 2.3^x + 1$

5. Translation of graphs

The translation of graphs means that an existing graph is moved to the right or the left by a specific number of units, and is translated upwards or downwards by a specific number of units.

The following rules apply

The **graph** is translated p units to the **right** $\Rightarrow x$ changes to $(x - p)$.
The **graph** is translated p units to the **left** $\Rightarrow x$ changes to $(x + p)$.

The graph is **translated** q units **upwards** $\Rightarrow y$ changes to $(y - q)$.
The graph is **translated** q units **downwards** $\Rightarrow y$ changes to $(y + q)$.

What will the equation of the new graph be if the graph of $y = x^2$ is moved 2 units to the left and translated 1 unit upwards.

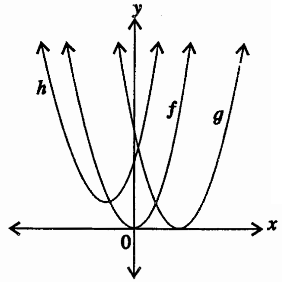
$y - 1 = (x + 2)^2$ [2 *left* → x *becomes* $(x + 2)$; 1 *upwards* → y *becomes* $(y - 1)$]

$\therefore\ y = (x + 2)^2 + 1$

Therefore, the graph of $y = (x + 2)^2 + 1$ can be sketched by moving the graph of $y = x^2$ two units to the left and one unit upwards.

Example 13

In the diagram $f(x) = x^2$, $g(x)$ is the translation of $f(x)$ 2 units to the right and $h(x)$ is the translation of $f(x)$ 1 unit to the left and 2 units upwards.



Determine the equations of $g(x)$ and $h(x)$.

Solution

$f(x) = x^2$

$\therefore\ y = x^2$

Equation of $g(x)$

$y = (x - 2)^2$ [*Translation* 2 *units to the right* : $\therefore\ x \to x - 2$]

Equation of $h(x)$

$y - 2 = (x + 1)^2$ [*Translation* 1 *unit to the left* : $x \to (x + 1)$; 2 *units upwards* : $y \to y - 2$]

$\therefore\ y = (x + 1)^2 + 2$

Example 14

14.1 Sketch the graph of $y = x^2 - 2x - 3$.

14.2 On the same set of axes, using translation, sketch the graph of $f(x) = (x + 2)^2 - 1$.

Solution

14.1 *y-intercept* : $y = -3$

x-intercept : *Let* $y = 0$.

$x^2 - 2x - 3 = 0$

$\therefore\ (x-3)(x+1) = 0$

$\therefore\ x - 3 = 0 \quad or \quad x + 1 = 0$

$\therefore\ x = 3 \qquad or \qquad x = -1$

Turning Point

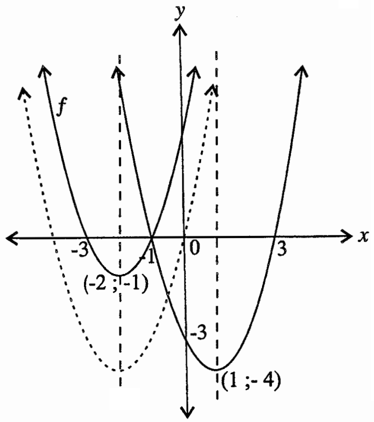
$x = -\dfrac{b}{2a} \qquad = -\dfrac{-2}{2(1)} \qquad = 1$

Substitute $x = 1$ *into equation*

$\therefore\ y = (1)^2 - 2(1) - 3$

$\quad = -4$

Turning Point $(1;-4)$



14.2 *Turning Point of* $f(x) = (-2;-1)$.

Therefore, a translation 3 *units to the left and* 3 *units upwards.*

First move graph 3 *units to the left.*
$\therefore$ *Turning Point* $(-2;-4)$ *and roots* $x = 0$ *and* $x = -4$.

Now move graph 3 *units upwards.*
$\therefore$ *Turning Point* $(-2;-1)$ *and roots* $x = -1$ *and* $x = -3$.

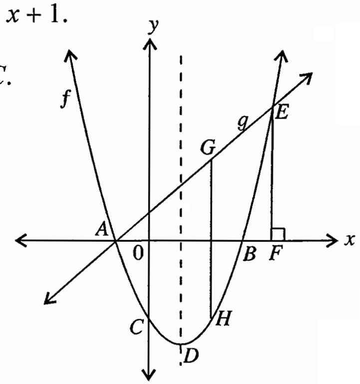
6. Deductions from sketch graphs

You must be able to make certian deductions from sketch graphs when the equation of the graph is given, i.e. determine coordinates, lengths of lines and the points of intersection of graphs .

Example 15

In the diagram $f(x) = x^2 - 4x - 5$ and $g(x) = x + 1$.

15.1 Calculate the lengths of OA, OB and OC.
15.2 Calculate the coordinates of D, the turning point of $f(x)$.
15.3 Calculate the length of EF.
15.4 Calculate the maximum length of GH.
15.5 What is the range of $f(x)$?
15.6 Give the coordinates of the turning point of $f(x+2)$ and describe the translation that took place.



Solution

15.1. *To calculate the lengths of OA, OB and OC, you first have to find the coordinates of A, B and C.*
A and B are the x-intercepts, therefore let $y = 0$.

$\therefore\ x^2 - 4x - 5 = 0$
$\therefore\ (x-5)(x+1) = 0$
$\therefore\ x = 5$ *or* $x = -1 \qquad A(-1;0) \quad B(5;0)$

Coordinates of $C(0;-5)$ $\qquad$ [*C is y-intercept* $\therefore\ x = 0$]

$\therefore$ *OA* = 1 *unit* $\quad$ *OB* = 5 *units* $\quad$ *OC* = 5 *units*

15.2 *Coordinates of Turning Point* : $x = \dfrac{-b}{2a} \qquad \therefore\ x = \dfrac{-(-4)}{2(1)}$

$\therefore\ x = \frac{4}{2}$

$\therefore\ x = 2$

Substitute $x = 2$ *into the equation* : $y = (2)^2 - 4(2) - 5$
$\therefore\ y = 4 - 8 - 5$
$\therefore\ y = -9$

$\therefore\ D(2;-9)$

15.3 *EF is parallel to the y-axis → length of EF = y-coordinate of E.*

First calculate the x-coordinate of E. E is the point of intersection of f(x) and g(x).

Let f(x) = g(x) and solve the equation.

$\therefore\ x^2 - 4x - 5 = x + 1$ $\quad$ $[f(x) = x^2 - 4x - 5$ *and* $g(x) = x + 1]$
$\therefore\ x^2 - 5x - 6 = 0$
$\therefore\ (x - 6)(x + 1) = 0$
$\therefore\ x = 6$ *or* $x = -1$ *but* $x > 0$ *at* E $\quad \therefore$ *At* $E: x = 6$

Let $x = 6$ *in* $g(x): y = x + 1$
$\therefore\ y = 6 + 1$
$\therefore\ y = 7$

$\therefore\ E\,(6;7)$

$\therefore$ *Length of EF = 7 units*

15.4 *GH is the difference between g(x) and f(x).*

$\therefore\ GH = g(x) - f(x)$
$\therefore\ GH = x + 1 - (x^2 - 4x - 5)$
$\therefore\ GH = x + 1 - x^2 + 4x + 5$
$\therefore\ GH = -x^2 + 5x + 6$ $\quad$ [*Now complete the square*]
$\therefore\ GH = -(x^2 - 5x - 6)$

$\therefore\ GH = -(x^2 - 5x + (-\frac{5}{2})^2 - (-\frac{5}{2})^2 - 6)$

$\therefore\ GH = -[(x - \frac{5}{2})^2 - \frac{25}{4} - 6]$

$\therefore\ GH = -[(x - \frac{5}{2})^2 - \frac{49}{4}]$

$\therefore\ GH = -(x - \frac{5}{2})^2 + \frac{49}{4}$ $\quad$ $[-\frac{49}{4} \times -1 = \frac{49}{4}]$

$\therefore$ *Maximum length of GH* $= \frac{49}{4} = 12¼$

15.5 *Range is the y-values. The graph has a turning point at (2;–9). Therefore, the minimum value is* -9.

$\therefore$ *Range* $= \{y;\ y \geq -9,\ y \in \mathbb{R}\}$.

15.5 *Turning Point of* $f(x + 2) = (0;-9)$ $\quad$ [*Graph moves 2 units to the left*]

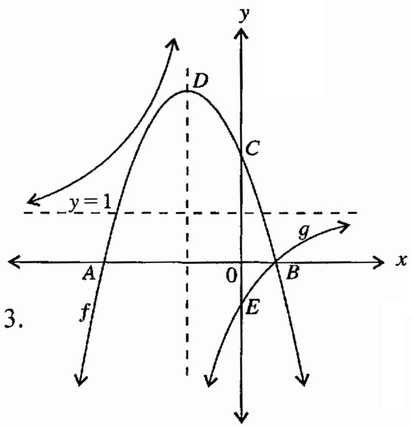
$\therefore$ *Translation of 2 units to the left.*

Example 16

The diagram represents the graphs of

$f(x) = -x^2 - 2x + 3$ and $g(x) = \dfrac{-2}{x + p} + q$.

D is the turning point of the parabola.



16.1 Determine the average gradient of $f(x)$ between the points $x = 2$ and $x = 3$.
16.2 Determine the values of *p* and *q*.
16.3 Hence, calculate the coordinates of *E*.
16.4 Give the domain and range of $g(x)$.
16.5 Calculate $g(-5)$

Solution

16.1 *Average gradient* $= \dfrac{\text{Change in } y}{\text{Change in } x}$ *or* $\dfrac{y_1 - y_2}{x_1 - x_2}$

So first find the values of y when x = 2 and x = 3.

$y = -(2)^2 - 2(2) + 3$ $\qquad$ $y = -(3)^2 - 2(3) + 3$
$\therefore\ y = -5$ $\qquad$ $\therefore\ y = -12$

$\therefore$ *Average gradient between* $(2,-5)$ *and* $(3;-12)$

$\therefore$ *Average gradient* $= \dfrac{-5 - (-12)}{2 - 3} = \dfrac{-5 + 12}{-1}$

$= \underline{-7}$

OR

$$\textit{Average gradient} = \frac{f(2) - f(3)}{2 - 3}$$

$$= \frac{[-(2)^2 - 2(2) + 3] - [-(3)^2 - 2(3) + 3]}{2 - 3}$$

$$= \frac{-5 - (-12)}{-1}$$

$$= \frac{-5 + 12}{-1} \qquad = -7$$

16.2 *The vertical asymptote passes through the turning point D. Therefore, calculate the x-coordinate of the turning point.*

$$x = -\frac{b}{2a}$$

$$\therefore\ x = -\frac{-2}{2(-1)}$$

$$\therefore\ x = -1$$

$\underline{p = 1}$ [*Vertical asymptote* : $x = -p$, $\therefore\ -p = -1$, $\therefore\ p = 1$]

$\underline{q = 1}$ [*Horizontal asymptote* : $y = q$]

16.3 *E is the y-intercept of g(x), the hyperbola.* $\therefore\ x = 0$

$$y = \frac{-2}{0 + 1} + 1 \qquad [p = 1 \textit{ and } q = 1.\ \textit{Let } x = 0\ ]$$

$$\therefore\ y = -2 + 1$$

$$\therefore\ y = -1 \qquad \underline{\therefore\ E(0;-1)}$$

16.4 *Domain* = *the x-values* = $\underline{\{x : x \in R,\ x \neq -1\}}$

Range = *the y-values* = $\underline{\{y : y \in R,\ y \neq 1\}}$

16.5 $g(5) = \frac{-2}{-5 + 1} + 1$ [*Substitute x with −5 in* $g(x) = \frac{-2}{x + 1} + 1$]

$$= \frac{-2}{-4} + 1$$

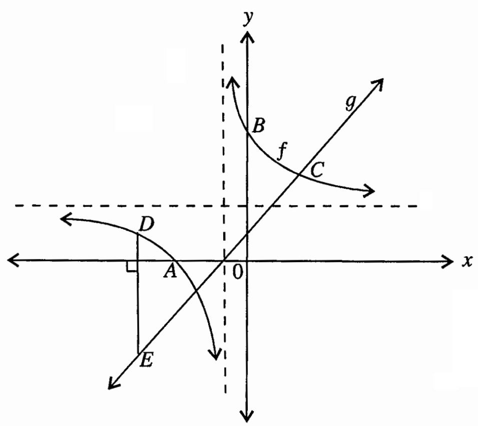
$$= \underline{\frac{3}{2}}$$

Example 17

Sketched are the graphs of

$$f(x) = \frac{3}{x + 1} + 2$$

and $g(x) = x + 1$.



17.1 Calculate the coordinates of A and B.
17.2 Determine the coordinates of C.
17.3 If $DE = 4$ units, determine the coordinates of E.
17.4 Determine the equation of the axis of symmetry of $f(x)$.
17.5 h is the graph of f shifted 2 units to the right and one unit downwards. Give the equation of $h(x)$.

Solution

17.1 *A is the x-intercept of f(x). Let y = 0.*

$$0 = \frac{3}{x + 1} + 2 \qquad [y = 0,\ \textit{determine } x\ ]$$

$$\therefore\ 0 = 3 + 2(x + 1) \qquad [\textit{Multiply by } (x + 1)\ ]$$

$$\therefore\ 0 = 3 + 2x + 2$$

$$\therefore\ -2x = 5 \qquad [\textit{Move } 2x \textit{ to left-hand side}\ ]$$

$$\therefore\ x = -\frac{5}{2} \qquad \underline{\therefore\ A(-\tfrac{5}{2};0)}$$

B is the y-intercept of f(x). Let x = 0.

$\therefore\ y = \frac{3}{0+1} + 2$ [*x = 0, determine y*]

$\therefore\ y = 3 + 2$

$\therefore\ y = 5$ $\therefore\ B(0;5)$

17.2 *C is the point of intersection of f(x) and g(x). Let f(x) = g(x).*

$\therefore\ \frac{3}{x+1} + 2 = x + 1$

$\therefore\ 3 + 2(x+1) = (x+1)(x+1)$ [*Multiply by LCD → (x + 1)*]

$\therefore\ 3 + 2x + 2 = x^2 + x + x + 1$ [*Remove brackets*]

$\therefore\ 0 = -3 - 2x - 2 + x^2 + x + x + 1$

$\therefore\ 0 = x^2 - 4$

$\therefore\ 0 = (x-2)(x+2)$ [*Factorise RHS*]

$\therefore\ x = 2$ *or* $x = -2$

At C, $x > 0$ $\therefore\ x = 2$

Let $x = 2$ *in* $y = x + 1$

$\therefore\ y = 2 + 1$

$\therefore\ y = 3$ $\therefore\ C(2;3)$

17.3 *DE is the difference between f(x) and g(x).*

$\therefore\ DE = f(x) - g(x)$

$\therefore\ DE = \frac{3}{x+1} + 2 - (x+1)$

$\therefore\ 4 = \frac{3}{x+1} + 2 - x - 1$ [*DE = 4, given*]

$\therefore\ 4(x+1) = 3 + 2(x+1) - x(x+1) - (x+1)$ [*Multiply by (x + 1)*]
$\therefore\ 4x + 4 = 3 + 2x + 2 - x^2 - x - x - 1$
$\therefore\ 4x + 4 - 3 - 2x - 2 + x^2 + x + x + 1 = 0$
$\therefore\ x^2 + 4x = 0$
$\therefore\ x(x+4) = 0$
$\therefore\ x = 0$ *or* $x = -4$

At E, $x < 0$ $\therefore\ x = -4$

E is a point on g(x). Therefore, let x = −4 in g(x).

$\therefore\ y = -4 + 1$
$\therefore\ y = -3$ $\therefore\ E(-4;-3)$

17.4 *Equation of axis of symmetry*

$y = (x+p) + q$ *or* $y = -(x+p) + q$
$\therefore\ y = (x+1) + 2$ $\therefore\ y = -(x+1) + 2$
$\therefore\ y = x + 1 + 2$ $\therefore\ y = -x - 1 + 2$
$\therefore\ y = x + 3$ $\therefore\ y = -x + 1$

$\therefore\ y = x + 3$ *or* $y = -x + 1$

17.5 1 *unit downwards* : $y \Rightarrow y + 1$. → 2 *units to the right* : $x \Rightarrow x - 2$

$\therefore\ y + 1 = \frac{3}{(x-2)+1} + 2$

$\therefore\ y = \frac{3}{x-1} + 2 - 1$

$\therefore\ h(x) = \frac{3}{x-1} + 1$

7. Graphs of the trigonometric functions

7.1 The graph of $y = a\sin x + q$, $y = a\cos x + q$ and $y = a\tan x + q$

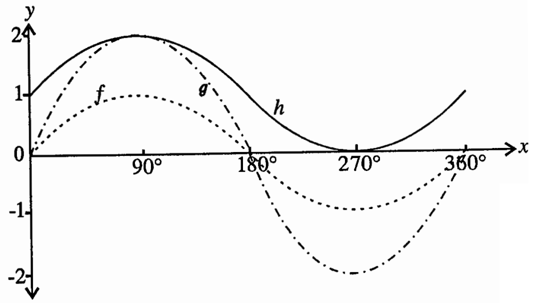
In Grade 10 you have sketched the graphs of $y = a\sin x + q$, $y = a\cos x + q$ and $y = a\tan x + q$

The diagram shows the graphs

of $f(x) = \sin x$,

$g(x) = 2\sin x$

and $h(x) = \sin x + 1$.

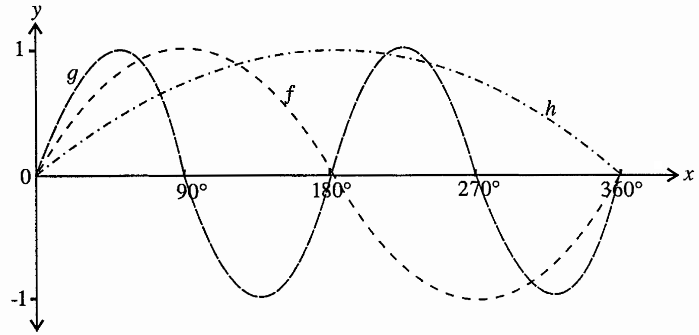


Effect of a : a is a vertical stretching or shrinking of the graph and has an effect on the amplitude and the range of the function.

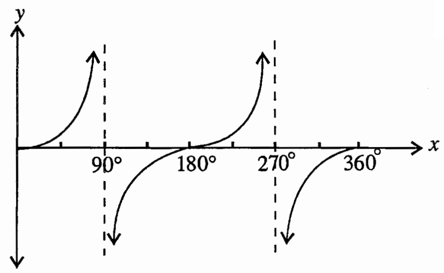
Effect of q : q is a vertical shift of the graph and therefore effects the range of the function.

7.2 The graphs of $y = \sin(kx)$; $y = \cos(kx)$ and $y = \tan(kx)$

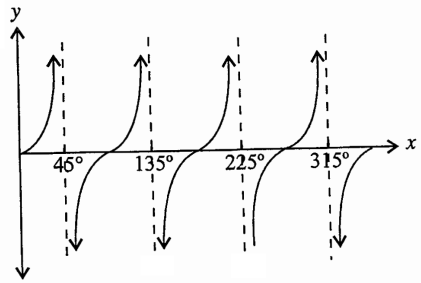
Sketched are the graphs of $f(x) = \sin x$, $(x) = \sin 2x$ and $h(x) = \sin \frac{1}{2}x$.



Take a look at the following sketches.



$y = \tan x$



$y = \tan 2x$

The effect of k : It is clear that k is a horizontal stretching or shrinking of the graph and therefore k has an effect on the period of the function.

The period of $y = \sin x$ and $y = \cos x$ is 360° while the period of $y = \tan x$ 180° is.

The period of $y = \sin(kx)$ and $y = \cos(kx)$ is $\frac{360°}{k}$ and the period of

$y = \tan(kx)$ is $\frac{180°}{k}$.

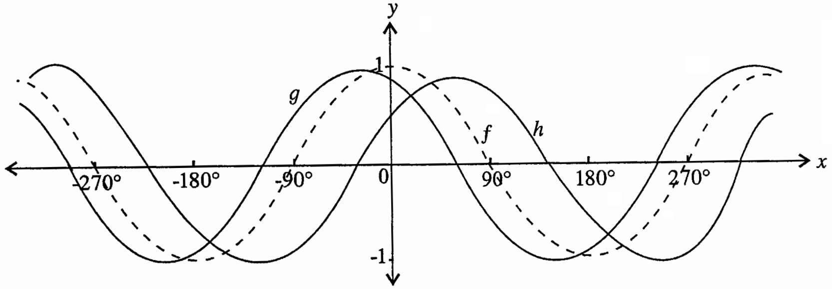
To sketch the graphs of the trigonometric functions, use your calculator to set up a table and sketch the graph by means of point-by-point plotting.

However it is important that you know the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ and can sketch the graphs. Know the x-intercepts and the turning points of the various functions for $x \in [0°; 360°]$.

This will make it easier to do translations as well as to determine the equations of the functions when sketches of graphs are given.

7.3 The graphs of $y = \sin(x+p)$, $y = \cos(x+p)$ and $y = \tan(x+p)$

Sketched are the graphs of $f(x) = \cos(x)$, $g(x) = \cos(x + 30°)$ and $h(x) = \cos(x - 60°)$.



The effect of p : $g(x)$ is the graph of $f(x)$ translated 30° to the left while $h(x)$ is the translation of $f(x)$ 60° to the right. Therefore : p is the horizontal shift of the graph.

7.4 The graphs of $y = a\sin k(x+p)$, $y = a\cos k(x+p)$ and $y = a\tan k(x+p)$

You already know that : a is a **vertical stretching or shrinking** of the graph,
k is a **horizontal stretching or shrinking** and
p is a **horizontal shift** of the graph.

Example 18

Sketch the graph of $y = 2\sin 2(x - 30°)$ for $x \in [-180°; 180°]$.

Solution

$k = 2 \rightarrow$ *Horizontal shrinking* $\rightarrow$ *Period* $= 360° \div 2 = 180°$.
$a = 2 \rightarrow$ *Vertical stretching* $\rightarrow$ *Maximum* $= 2$, *minimum* $= -2$.
$p = -30° \rightarrow$ *Horizontal shift* 30° *to the right.*

You can sketch the graph of $y = \sin x$ and then do the translations,

or

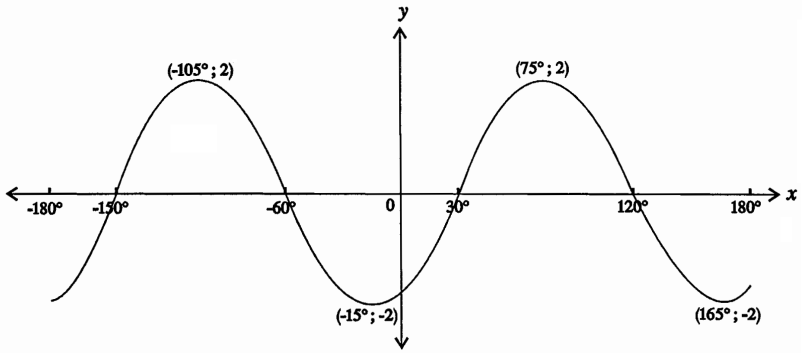
Set up a table.

x	−180°	−165°	−150°	−135°	−120°	−105°	−90°	−75°
$f(x)$	−1,73	−1	0	1	1.73	2	1.73	1

−60°	−45°	−30°	−15°	0°	15°	30°	45°	60°
0	−1	−1.73	−2	−1.73	−1	0	1	1.73

75°	90°	105°	120°	135°	150°	165°	180°
2	1.73	1	0	−1	−1.73	−2	−1.73

Now sketch the graph by means of point-by-point plotting.



Very small intervals are used in order to obtain the values of the turning points.

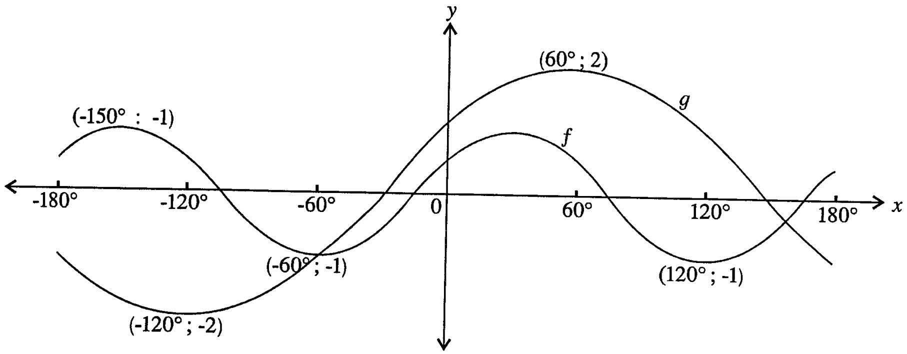
In the graph of $y = 2\sin 2(x - 30°)$ 3 parameters are used , $a(2)$, $k(2)$ and $p(30°)$. In the examination no more than two parameters at a time will be asked simultaneously, eg. $y = \sin k(x+p)$ or $y = a\sin(x+p)$ or $y = a\sin(kx)$ or $y = \sin(x+p) + q$

Example 19

On the same set of axes, sketch the graphs of $f(x) = \cos(2x - 60°)$ and $g(x) = 2\sin(x + 30°)$ for $x \in [-180°; 180°]$.

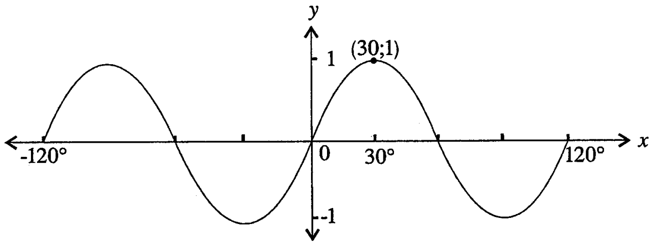
Solution

Use your calculator to set up tables and then plot the points.



Example 20

Sketched are the graphs of $y = \sin ax$ for $x \in [-120°; 120°]$.



20.1 Determine the value of a.
20.2 What is the range of $y = \sin ax + 1$

Solution

20.1 $\sin a(30°) = 1$ [$(30°; 1)$ *is a point on the graph*]

$\therefore\ a(30°) = 90°$ [$\sin 90° = 1$ | *shift* | sin | 1 | = |]

$\underline{\therefore\ a = 3.}$ [$90° \div 30° = 3$]

[*Or : The period of* $\sin ax = 120°$: $\therefore\ \frac{360°}{a} = 120°$ $\therefore\ a = 3$]

20.2 *Range of* $y = \sin ax$ *is* $-1 \leq y \leq 1$

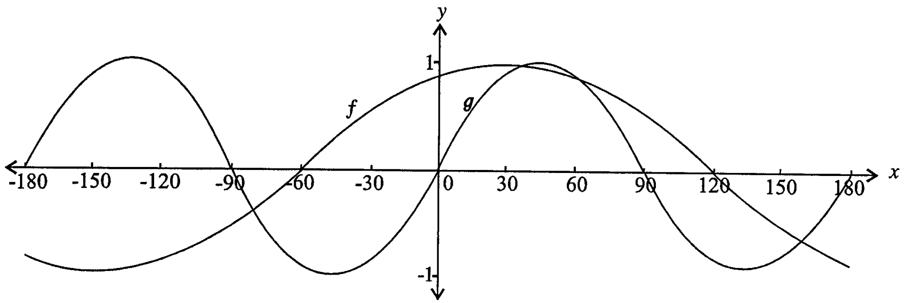
$\therefore$ *Range of* $y = \sin ax + 1$ *is* $-1 + 1 \leq y \leq 1 + 1$
$\underline{\therefore\ 0 \leq y \leq 2}$

Example 21

21.1 On the same set of axes, draw sketch graphs of $f(x) = \cos(x - 30°)$ and $g(x) = \sin 2x$ for $x \in [-180°; 180°]$.
21.2 For which values of x in the interval $x \in [0°; 90°]$ is $f(x) = g(x)$.

Solution :

21.1 *Set up a table or sketch using translations.*



21.2 $\cos(x - 30°) = \sin 2x$
$\therefore\ \cos(x - 30°) = \cos(90° - 2x)$

$\therefore\ x - 30° = 90° - 2x$ *or* $x - 30° = 360° - (90° - 2x)$
$\therefore\ x + 2x = 90° + 30° + k.360°$ $\therefore\ x - 30° = 270° + 2x + k.360°$
$\therefore\ 3x = 120° + k.360°$ $\therefore\ x - 2x = 270° + 30° + k.360°$
$\therefore\ x = 40° + k.120°$ $\therefore\ -x = 300° + k.360°$
$\therefore\ x = -300° + k.360°$

Let $k = 1$: $x = -300° + (1)360° = 60°$.

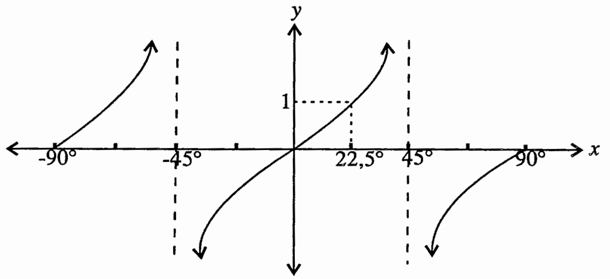
$\therefore\ f(x) = g(x)$ *for* $x \in [0°; 90°]$ *when* $\underline{x = 40°\ \ or\ \ x = 60°.}$

Example 22

Sketch the graph of $y = \tan 2x$ $\{-90° \leqslant x \leqslant 90°\}$

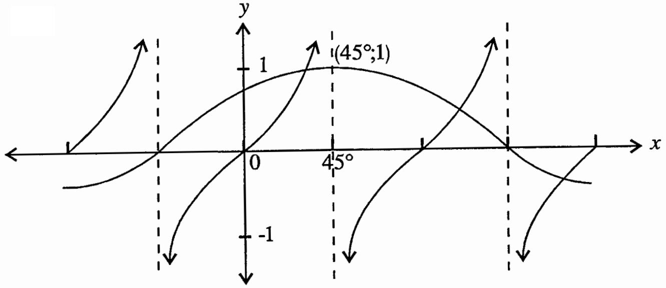
Solution

Asymptotes at $x = 45° + k.90°$. *Use intervals of* 22,5° *on x-axis.*



Example 23

Sketched are the graphs of $f(x) = \tan ax$ and $g(x) = \cos(x + p°)$.
Determine the values of a and p.



Solution

Asymptote of $\tan x$ *is* $x = 90°$.

$\therefore\ a(45°) = 90°$ [*The asymptote of* $\tan ax$ *is* $x = 45°$]

$\therefore\ a = 2$ [$90 \div 45 = 2$]

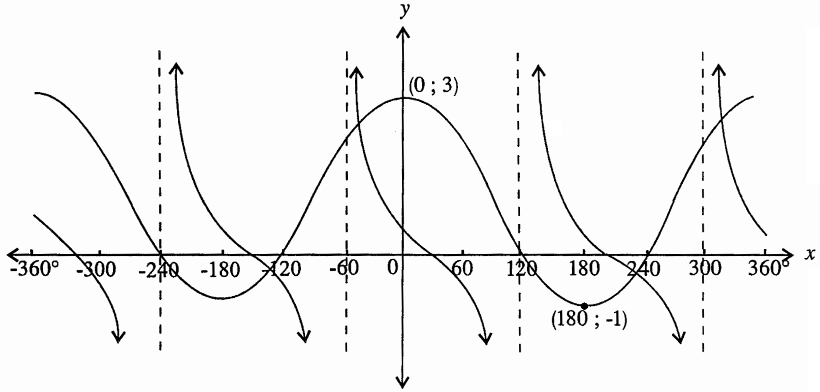
$\cos(45° + p) = 1$ [(45°; 1) *point on* $g(x)$.]

$\therefore\ 45° + p = 0°$ [shift cos 1 =]

$\therefore\ p = -45°$

Example 24

The diagram shows the graphs of $f(x) = a\tan(x + p)$ and $g(x) = a\cos x + q$ for $x \in [-360°; 360°]$. Determine the equations of $f(x)$ and $g(x)$.



Solution

Asymptote of $a\tan(x + p)$: $x = 120°$ → *Asymptote of* $\tan x$: $x = 90°$

$\therefore$ *f(x) is horizontal shift of* $y = \tan x$ 30° *to the right and a reflection of* $y = \tan x$ *in the x-axis.*

$\therefore\ f(x) = -\tan(x - 30°)$

Amplitude of $g(x) = 2$ → *Amplitude of* $\cos x = 1$.

$\therefore\ a = 2$

Maximum value of $2\cos x = 2$

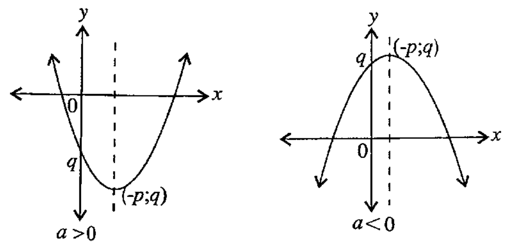
Maximum value of $a\cos x + q = 3$ → *Vertical shift* 1 *unit upwards*

$\therefore\ q = 1$

$\therefore\ g(x) = 2\cos x + 1$

SUMMARY

The Parabola : $y = a(x+p)^2 + q$



Coordinates of Turning Point : $(-p;q)$

Axis of symmertry : $x = -p$

The parabola do not have any asymptotes.

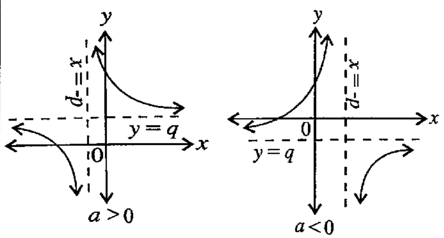
$a > 0$ → the function has a minimum value.

$a < 0$ → the function has a maximum value.

The maximum or minimum value = q.

Or for Turning Point of parabola : Let $x = -\dfrac{b}{2a}$ and substitute this value into equation to find the y-value of the turning point.

The Hyperbola : $y = \dfrac{a}{x+p} + q$



Horizontal asymptote : $y = q$

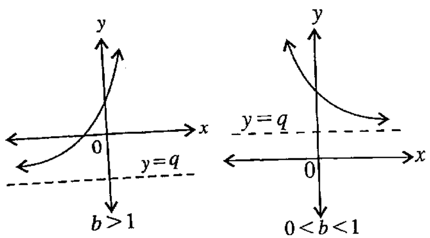
Vertical asymptote : $x = -p$

Axes of symmetry : $y = \pm(x+p) + q$

$a > 0$ → graph in 1st and 3rd quadrant

$a < 0$ → grafiek in 2de en 4de kwadrant

The exponential function : $y = ab^{x+p} + q$



Horizontal asymptote : $y = q$

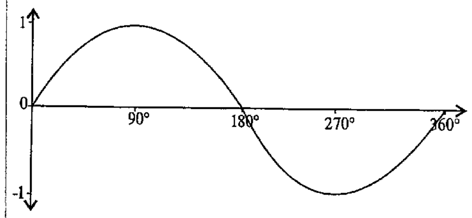
No vertical asymptote.

$b > 1$ → graph is ascending

$0 < b < 1$ → graph is descending

The graph do not have an axis of symmetry.

The graph of $y = \sin x \quad x \in [0°;360°]$

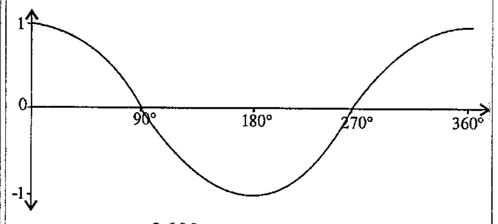


The period = $360°$

The amplitude = 1

Maximum value = 1 Minimum value = -1

The graph of $y = \cos x \quad x \in [0°;360°]$

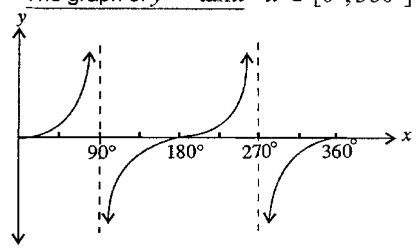


The period = $360°$

The amplitude = 1

Maximum value = 1 Minimum value = -1

The graph of $y = \tan x \quad x \in [0°;360°]$



The period = $180°$

Asymptotes : $x = 90°$ and $x = 270°$

No maximum value or minimum value

The effect of the parameters a, p, q and k

a is the vertical stretching or shrinking of the graph.
p is the horizontal shift of the graph.
q is the vertical shift of the graph.
To determine the x-intercepts, let $y = 0$ and to determine the y-intercept, let $x = 0$.
k is a horizontal stretching or shrinking of the graph and has an effect on the period of the function.